Kuroda's Identity for Mixed Lumped and Distributed Circuits and Their Application to Nonuniform Transmission Lines

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Abstract—**Kuroda's identities, which are used in analysis and synthesis of distributed transmission line circuits, may be applied to mixed lumped and distributed circuits. It is shown that circuits consisting of cascade connections of lumped reactances and uniform transmission lines are equivalent to circuits consisting of a cascade connection of nonuniform transmission lines, lumped reactances, and ideal transformers. Moreover, by using these equivalent transformations, network functions of some nonuniform transmission lines can be derived exactly.**

I. Introduction

It is well known that Kuroda's identities are very significant in analysis and synthesis of distributed transmission line circuits [1]–[8]. Many design methods for filters, matching sections, and other circuits have been obtained using Kuroda's identities. On the other hand, many articles have discussed the analysis and synthesis of nonuniform transmission lines [9]–[15], and it is well known that it is quite difficult to find the exact network functions of general nonuniform transmission lines from the telegrapher's equation.

In this paper, we apply Kuroda's identities to mixed lumped and distributed circuits and show that the equivalent circuits of a class of nonuniform transmission lines can be represented with mixed lumped and distributed circuits. Namely, Kuroda's identities are applied to circuits consisting of a cascade connection of lumped reactance elements and unit element (UE) as the limit case. Then, we can show that a class of nonuniform transmission lines, whose characteristic impedance distributions obey the binomial form, can be represented with circuits consisting of a cascade connection of lumped reactance elements, an UE and an ideal transformer. By repeating a similar procedure, it can be shown that the equivalent circuits of some nonuniform transmission lines are also expressed as mixed lumped and distributed circuits. This method may be useful to find exact solutions for nonuniform transmission lines. Finally, we apply these nonuniform transmission lines to impedance transformers.

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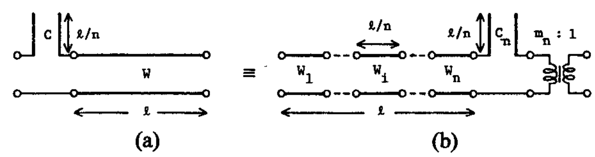


Fig. 1. Kuroda's identity for the series single open stub.

II. Kuroda's Identities for Mixed Lumped and Distributed Circuits

Kuroda's identities can be applied n-times (n:integer) to the circuit shown in Fig. 1 (a), where line length and the characteristic admittance of the single open stub are l/n and C, respectively, and line length and the characteristic impedance of the lossless uniform transmission line (UE) are l and W. The transformed circuit is one consisting of a cascade connection of UE's with line length l/n, a single open stub and an ideal transformer as shown in Fig. 1(b). The element values of the transformed circuit are given as follows:

$$W_i = W\frac{(CW+i-1)(CW+i)}{(CW)^2} \qquad (i=1,2,\cdots,n) \tag{1}$$

$$C_n = \frac{C^2W}{n+CW} \tag{2}$$

$$m_n = \frac{n+CW}{CW}. \tag{3}$$

Set

$$C = nC_0. \tag{4}$$

By substituting (4) in (1)–(3), we get

$$W_i = W\frac{\left(C_0W+\frac{i-1}{n}\right)\left(C_0W+\frac{i}{n}\right)}{(C_0W)^2} \qquad (i=1,2,\cdots,n) \tag{5}$$

$$C_n = \frac{nC_0^2W}{1+C_0W} \tag{6}$$

$$m_n = \frac{1+C_0W}{C_0W} \equiv m. \tag{7}$$

The characteristic impedances W_i of the UE's are discrete values increasing monotonically. Here, we define the dis-

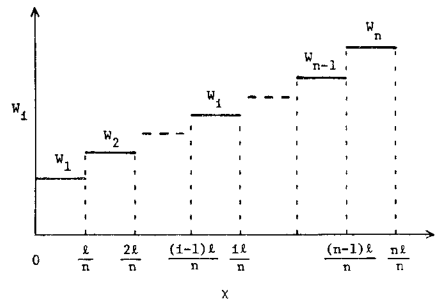


Fig. 2. The coordinates of the distance χ.

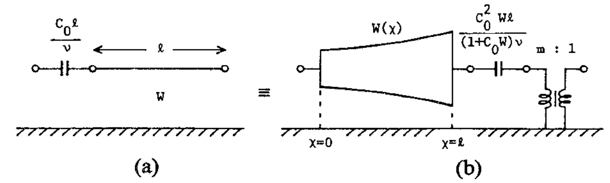


Fig. 3. Application of Kuroda's identity to the mixed lumped and distributed circuit.

crete coordinates of the distance χ as shown in Fig. 2, then the coordinates χ of the ith UE of the transformed circuit is given as follows:

$$\chi = \frac{i}{n} l. \tag{8}$$

By substituting (8) in (5) and proceeding to the limit $n \to \infty$ in such a way that $C \to \infty$ but C_0 remains finite, then we obtain

$$\lim_{n\to\infty} W_i = W\left(1 + \frac{1}{C_0 W}\frac{\chi}{l}\right)^2 \equiv W(\chi). \tag{9}$$

$W(\chi)$ is the characteristic impedance distribution of the parabolic tapered transmission line. Namely, if Kuroda's identity are applied a number of times ($n \to \infty$), the characteristic impedance distribution of the transformed circuit turns from a discrete function of the distance χ into a continuous function. Under this condition, the admittance y and Y of the single open stubs shown in Fig. 1(a) and (b), respectively, yield

$$\lim_{n\to\infty} y = \lim_{n\to\infty}\left[jnC_0 \tan\frac{\beta l}{n}\right] = jC_0\beta l = j\omega\frac{C_0 l}{v} \tag{10}$$

$$\lim_{n\to\infty} Y = \lim_{n\to\infty}\left[j\frac{nC_0^2 W}{1+C_0W}\tan\frac{\beta l}{n}\right]$$

$$= j\frac{C_0^2 W}{1+C_0 W}\beta l = j\omega\frac{C_0^2 Wl}{(1+C_0W)v} \tag{11}$$

where β is the phase constant, ω is the angular frequency, and v is the velocity of light. The single open stubs become lumped capacitors. Thus when Kuroda's identities are applied to the mixed lumped and distributed circuit shown in Fig. 3(a), the circuit of Fig. 3(b) consisting of a cascade connection of a parabolic tapered transmission line, a series lumped capacitor and an ideal transformer is obtained as an exact equivalent circuit. It may be shown also that the equivalent circuit of the parabolic tapered transmission line can be expressed as the mixed lumped and distributed circuit shown in Fig. 4(b).

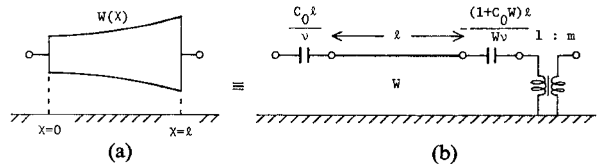


Fig. 4 Equivalent circuit of the parabolic tapered transmission line.

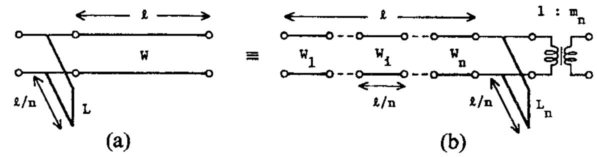


Fig. 5. Kuroda's identity for the shunt single short stub.

In the same manner, we can apply Kuroda's identity to the circuit shown in Fig. 5(a) n-times and obtain the circuit shown in Fig. 5(b), where L and L_n are the characteristic impedances of the single short stubs whose line lengths are l/n. The element values of the transformed circuit are given as follows:

$$W_i = \frac{L^2 W}{\{L+(i-1)W\}(L+iW)} \qquad (i=1,2,\cdots,n) \tag{12}$$

$$L_n = \frac{L^2}{L+nW} \tag{13}$$

$$m_n = \frac{L+nW}{L}. \tag{14}$$

We set

$$L = nL_0. \tag{15}$$

By substituting (8) and (15) in (12) and allowing n to approach infinity, we obtain the following characteristic impedance distribution:

$$\lim_{n\to\infty} W_i = \frac{W}{\left(1+\dfrac{W}{L_0}\dfrac{\chi}{l}\right)^2} \equiv W'(\chi). \tag{16}$$

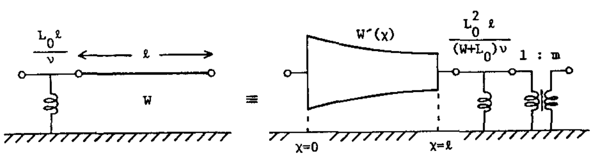
The single short stubs become lumped inductances with impedances given as follows:

$$\lim_{n\to\infty} z = \lim_{n\to\infty}\left[jnL_0 \tan\frac{\beta l}{n}\right] = jL_0\beta l = j\omega\frac{L_0 l}{v} \tag{17}$$

$$\lim_{n\to\infty} Z = \lim_{n\to\infty}\left[j\frac{nL_0^2}{L_0+W}\tan\frac{\beta l}{n}\right]$$

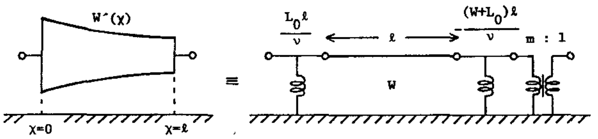
$$= j\frac{L_0^2}{L_0+W}\beta l = j\omega\frac{L_0^2 l}{(L_0+W)v}. \tag{18}$$

Namely, Kuroda's identity can be applied to the mixed lumped and distributed circuit shown in Fig. 6(a) and the transformed circuit is given in Fig. 6(b). The equivalent circuit of the nonuniform transmission line, whose characteristic impedance distribution is given in (16), is ex-



(a) (b)

Fig. 6. Application of Kuroda's identity to the mixed lumped and distributed circuit.



(a) (b)

Fig. 7. The nonuniform transmission line representing the reciprocal of the quadratic function and its equivalent circuit.

pressed by the mixed lumped and distributed circuit shown in Fig. 7 (b).

III. Kuroda's Identities for Mixed Lumped and Nonuniform Distributed Circuits

A. Second Order Binomial Form Nonuniform Transmission Lines

We apply Kuroda's identity to the circuit shown in Fig. 8(a) n-times, then obtain the circuit shown in Fig. 8(b). The values of the characteristic impedances W_i of the circuit shown in Fig. 8(a) are given in (1) and the element values of the transformed circuit are given as follows:

$$k_j = 1 + \frac{1}{L}\sum_{i=1}^{j} W_i = 1 + \frac{1}{LC^2W}\left\{\frac{j}{6}(j+1)(2j+1) + \frac{j}{2}(j+1)(2CW-1) + jCW(CW-1)\right\} \quad (j=1,2,\cdots,n) \tag{19}$$

$$Z_j = \frac{W_j}{k_{j-1}k_j} \qquad (j=1,2,\cdots,n) \tag{20}$$

$$L_n = \frac{L}{k_n}. \tag{21}$$

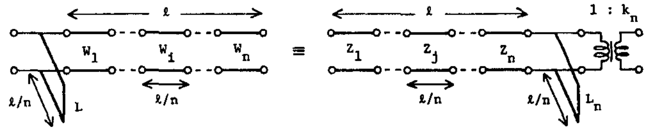
We set

$$L = nL_0 \qquad C = nC_0. \tag{22}$$

By substituting (8) and (22) in (20) and taking the limit $n \to \infty$ as previously, we obtain

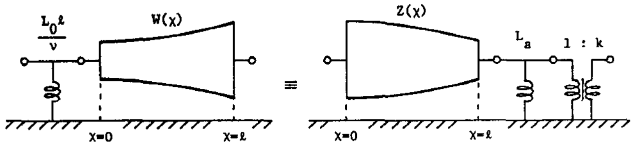
$$\lim_{n\to\infty} Z_j = \frac{W\left(1 + \frac{1}{C_0W}\frac{\chi}{l}\right)^2}{\left\{1 + \frac{W}{L_0}\left(\frac{\chi}{l}\right) + \frac{1}{L_0C_0}\left(\frac{\chi}{l}\right)^2 + \frac{1}{3L_0C_0^2W}\left(\frac{\chi}{l}\right)^3\right\}^2} \equiv Z(\chi). \tag{23}$$

When n approaches infinity, the circuit shown in Fig. 8(a) becomes one constructed as a cascade connection of a shunt lumped inductance and the parabolic tapered transmission line shown in Fig. 9(a), where $W(\chi)$ is the same as (9), and the circuit shown in Fig. 8(b) becomes one in Fig. 9(b). In Fig. 9(b), $Z(\chi)$ is the characteristic impedance distribution of the nonuniform transmission line and is given in (23), where the transformation ratio k and a lumped inductance L_a are given as follows:



(a) (b)

Fig. 8. Kuroda's identity for the shunt single short stub.



(a) (b)

Fig. 9. Application of Kuroda's identity to the mixed lumped and nonuniform-distributed circuit.

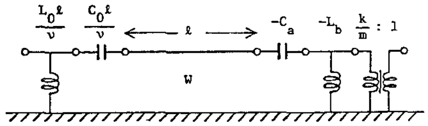


Fig. 10. The equivalent circuit of the nonuniform transmission line whose characteristic impedance distribution is given in (23).

$$k = 1 + \frac{W}{L_0} + \frac{1}{L_0C_0} + \frac{1}{3L_0C_0^2W} \tag{24}$$

$$L_a = \frac{1}{k}\frac{L_0 l}{v}. \tag{25}$$

The equivalent circuit of the nonuniform transmission line, whose characteristic impedance distribution is given by $Z(\chi)$, is expressed as the mixed lumped and distributed circuit shown in Fig. 10, where

$$m = 1 + \frac{1}{C_0W} \qquad C_a = m\frac{C_0 l}{v} \qquad L_b = \frac{k}{m^2}\frac{L_0 l}{v}. \tag{26}$$

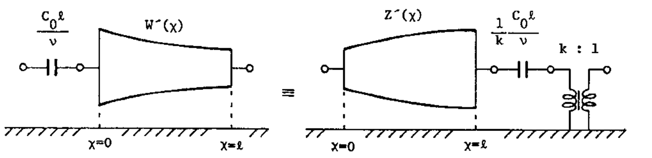
If the relation between the lumped capacitor C_0l/v and the lumped inductance L_0l/v is given by

$$\frac{3}{W}\frac{L_0 l}{v} = W\frac{C_0 l}{v} \tag{27}$$

the characteristic impedance distribution $Z(\chi)$ of (23) becomes equal to the 4th order binomial form

$$Z(\chi) = \frac{W}{\left(1 + \frac{1}{C_0W}\frac{\chi}{l}\right)^4}. \tag{28}$$

It is quite difficult to solve the telegrapher's equation of the nonuniform transmission line with $Z(\chi)$ in (23). But, by using Kuroda's identities described here, the exact network function can be easily obtained from the equivalent circuit shown in Fig. 10.



(a) (b)

Fig. 11. Kuroda's identity for the mixed lumped and nonuniform-distributed circuit.

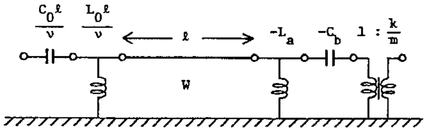
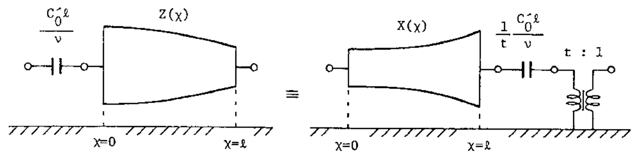


Fig. 12. The equivalent circuit of the nonuniform transmission line whose characteristic impedance distribution is given in (29).



(a) (b)

Fig. 13. Kuroda's identity for the mixed lumped and nonuniform-distributed circuit.

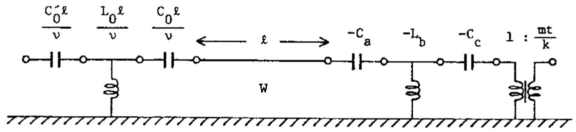


Fig. 14. The equivalent circuit of the nonuniform transmission line whose characteristic impedance distribution is given in (34).

Similarly, Kuroda's identity can be applied to a circuit consisting of a cascade connection of a series lumped capacitor and the nonuniform transmission line whose characteristic impedance distribution is given in (16) (see Fig. 11(a)). The transformed circuit consists of a cascade connection of a nonuniform transmission line, whose characteristic impedance distribution is $Z'(\chi)$, a series lumped capacitor, and an ideal transformer shown in Fig. 11(b), where

$$Z'(\chi)=\frac{W\left\{1+\frac{1}{C_0W}\frac{\chi}{l}+\frac{1}{L_0C_0}\left(\frac{\chi}{l}\right)^2+\frac{W}{3L_0^2C_0}\left(\frac{\chi}{l}\right)^3\right\}^2}{\left(1+\frac{W}{L_0}\frac{\chi}{l}\right)^2} \tag{29}$$

and

$$k=1+\frac{1}{C_0W}+\frac{1}{L_0C_0}+\frac{W}{3L_0^2C_0}. \tag{30}$$

Therefore, the equivalent circuit of this nonuniform transmission line with $Z'(\chi)$ is expressed by the mixed lumped and distributed circuit shown in Fig. 12, where

$$m=1+\frac{W}{L_0} \qquad L_a=m\frac{L_0l}{v} \qquad C_b=\frac{k}{m^2}\frac{C_0l}{v}. \tag{31}$$

Additionally, if the following relation is satisfied:

$$3W\frac{C_0l}{v}=\frac{1}{W}\frac{L_0l}{v} \tag{32}$$

then the characteristic impedance distribution $Z'(\chi)$ becomes the 4th order binomial form

$$Z'(\chi)=W\left(1+\frac{W}{L_0}\frac{\chi}{l}\right)^4. \tag{33}$$

B. Nonuniform Transmission Lines with $Z(\chi)$ and $Z'(\chi)$

We may apply Kuroda's identity to a circuit consisting of a cascade connection of a series lumped capacitor and a nonuniform transmission line whose characteristic impedance distribution $Z(\chi)$ is given in (23) (see Fig. 13(a)). The transformed circuit is given in Fig. 13(b), where a characteristic impedance distribution $X(\chi)$ of a nonuniform transmission line and a transformation ratio t are given as follows:

$$X(\chi)=W\left[1+\frac{1}{W}\left(\frac{1}{C_0}+\frac{1}{C_0'}\right)\frac{\chi}{l}+\frac{1}{L_0C_0'}\left(\frac{\chi}{l}\right)^2+\frac{1}{3L_0C_0'}\left(\frac{1}{C_0W}+\frac{W}{L_0}\right)\left(\frac{\chi}{l}\right)^3+\frac{1}{3C_0L_0^2C_0'}\left(\frac{\chi}{l}\right)^4+\frac{2}{15C_0^2L_0^2C_0'W}\left(\frac{\chi}{l}\right)^5+\frac{1}{45C_0^3L_0^2C_0'W^2}\left(\frac{\chi}{l}\right)^6\right]^2 \Big/ \left[1+\frac{W}{L_0}\frac{\chi}{l}+\frac{1}{C_0L_0}\left(\frac{\chi}{l}\right)^2+\frac{1}{3C_0^2L_0W}\left(\frac{\chi}{l}\right)^3\right]^2 \tag{34}$$

$$t=1+\frac{1}{C_0'W(1+C_0W)}\left[C_0W+\frac{W}{3L_0}(1+3C_0W)+\frac{1}{45(C_0L_0)^2}\left\{1+6C_0W+15(C_0W)^2+15(C_0W)^3\right\}\right]. \tag{35}$$

Therefore, the equivalent circuit of the nonuniform transmission line whose characteristic impedance distribution is given in (34) is equivalent to the mixed lumped and distributed circuit shown in Fig. 14, where

$$m=1+\frac{1}{C_0W} \qquad k=1+\frac{W}{L_0}+\frac{1}{C_0L_0}+\frac{1}{3C_0^2L_0W} \tag{36}$$

and

$$C_a=m\frac{C_0l}{v} \qquad L_b=\frac{k}{m^2}\frac{L_0l}{v} \qquad C_c=\frac{m^2t}{k^2}\frac{C_0'l}{v}. \tag{37}$$

In the special case where the following relation is satisfied:

$$5W\frac{C_0'l}{v}=\frac{3}{W}\frac{L_0l}{v}=W\frac{C_0l}{v} \tag{38}$$

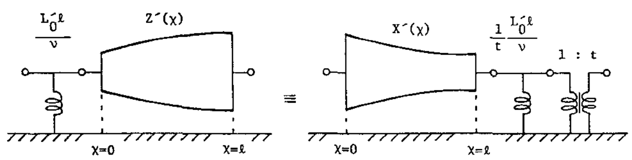


Fig. 15. Kuroda's identity for the mixed lumped and nonuniform-distributed circuit.

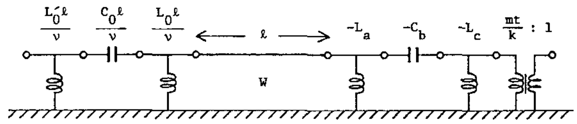


Fig. 16. The equivalent circuit of the nonuniform transmission line whose characteristic impedance distribution is given in (40).

then the characteristic impedance distribution $X(\chi)$ of (34) becomes equal to the 6th order binomial form

$$X(\chi)=W\left(1+\frac{1}{C_0W}\frac{\chi}{l}\right)^6. \tag{39}$$

Similarly, we obtain the circuit shown in Fig. 15(b) as a equivalent circuit of Fig. 15(a), where the characteristic impedance distribution $Z'(\chi)$ is given in (29), and the characteristic impedance distribution $X'(\chi)$ and a transformation ratio t are given as follows:

$$\begin{aligned}X'(\chi)=W&\left[1+\frac{1}{C_0W}\frac{\chi}{l}+\frac{1}{L_0C_0}\left(\frac{\chi}{l}\right)^2+\frac{W}{3L_0^2C_0}\left(\frac{\chi}{l}\right)^3\right]^2\Big/\\ &\left[1+W\left(\frac{1}{L_0}+\frac{1}{L_0'}\right)\frac{\chi}{l}+\frac{1}{C_0L_0'}\left(\frac{\chi}{l}\right)^2\right.\\ &+\frac{1}{3C_0L_0'}\left(\frac{W}{L_0}+\frac{1}{C_0W}\right)\left(\frac{\chi}{l}\right)^3+\frac{1}{3L_0C_0^2L_0'}\left(\frac{\chi}{l}\right)^4\\ &\left.+\frac{2W}{15L_0^2C_0^2L_0'}\left(\frac{\chi}{l}\right)^5+\frac{W^2}{45L_0^3C_0^2L_0'}\left(\frac{\chi}{l}\right)^6\right]^2\end{aligned} \tag{40}$$

$$\begin{aligned}t=1+\frac{W^2}{L_0'(W+L_0)}&\left[\frac{L_0'}{W}+\frac{1}{3C_0W}\left(1+3\frac{L_0}{W}\right)\right.\\ &\left.+\frac{1}{45(L_0C_0)^2}\left\{1+6\frac{L_0}{W}+15\left(\frac{L_0}{W}\right)^2+15\left(\frac{L_0}{W}\right)^3\right\}\right].\end{aligned} \tag{41}$$

The equivalent circuit of the nonuniform transmission line defined as $X'(\chi)$ is equivalent to the one shown in Fig. 16, where

$$m=1+\frac{W}{L_0}\qquad k=1+\frac{1}{C_0W}+\frac{1}{L_0C_0}+\frac{W}{3L_0^2C_0} \tag{42}$$

and

$$L_a=m\frac{L_0l}{v}\qquad C_b=\frac{k}{m^2}\frac{C_0l}{v}\qquad L_c=\frac{m^2t}{k^2}\frac{L_0'l}{v}. \tag{43}$$

In the special case where the following relation is applicable

$$5\frac{L_0'}{W}=3C_0W=\frac{L_0}{W} \tag{44}$$

then (40) simplifies as follows:

$$X'(\chi)=\frac{W}{\left(1+\frac{W}{L_0}\frac{\chi}{l}\right)^6}. \tag{45}$$

We can continue this procedure for mixed lumped and distributed circuits indefinitely. Therefore, it can be stated that the equivalent circuits of a class of nonuniform transmission lines are given by mixed lumped and distributed circuits, and network functions of these nonuniform transmission lines may be easily obtained from the equivalent circuits.

IV. Application to Impedance Transformers

We may apply the nonuniform transmission line theory described in the preceding sections to impedance transformers and compare the transmission characteristics with the exponential transmission line. The characteristic impedance distribution $E(\chi)$ of the exponential transmission line is expressed as

$$E(\chi)=W\exp\left(\delta\frac{\chi}{l}\right) \tag{46}$$

where δ is a taper coefficient.

The terminations at the input and output ports are assumed as 1 and R, respectively. We assume that the characteristic impedances of these nonuniform transmission lines at the input and output ports are 1 and R, respectively. Furthermore, for simplicity, the series lumped capacitor C_0l/v and the shunt lumped inductance L_0l/v shown in Fig. 12 are set to be equal. Then $W(\chi)$ in (9) and $Z(\chi)$ in (29) simplify as follows:

$$W(\chi)=\left(1+\frac{1}{h}\frac{\chi}{l}\right)^2 \tag{47}$$

$$Z(\chi)=\left[1+\frac{1}{h}\frac{\chi}{l}+\frac{1}{h^2}\left(\frac{\chi}{l}\right)^2+\frac{1}{3h^3}\left(\frac{\chi}{l}\right)^3\right]^2\Big/\left(1+\frac{1}{h}\frac{\chi}{l}\right)^2 \tag{48}$$

where

$$h=C_0=L_0. \tag{49}$$

Let the transformation ratio R be

$$R=10. \tag{50}$$

Then the taper coefficient δ in (46) and h in (47) are given

$$\delta=2.303$$
$$h=0.4625 \tag{51}$$

and the value of h in (48) is given as

$$h=0.5080. \tag{52}$$

In Fig. 17, we show the voltage standing wave ratio (VSWR) of these nonuniform transmission lines and opti-

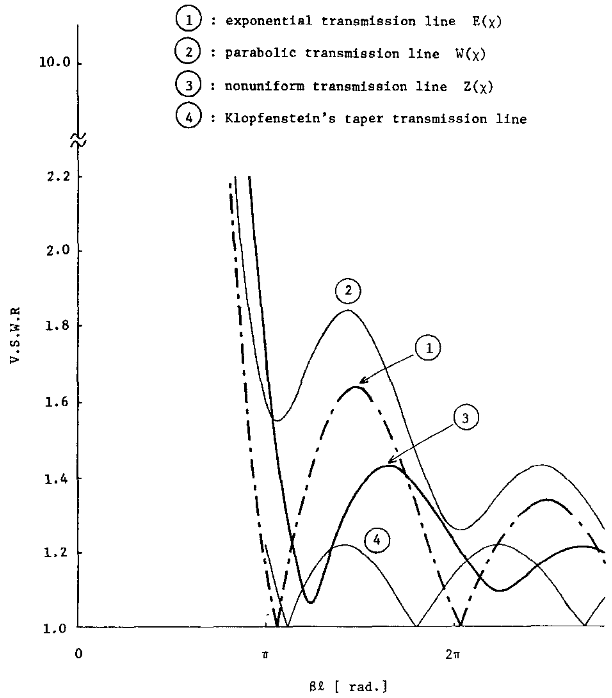


Fig. 17. The characteristics of VSWR of matching section.

mum taper transmission line obtained by Klopfenstein [10]. It is seen that the nonuniform transmission line defined by $Z(\chi)$ in (48) has broader band matching characteristics than the exponential transmission line.

V. Conclusion

We have applied Kuroda's identity to circuits consisting of a cascade connection of a single stub with line length l/n and an UE with line length l. By considering the limit case ($n \rightarrow$infinity), Kuroda's identities are extended to the case of mixed lumped and distributed circuits. The transformed circuits consist of a cascade connection of a nonuniform transmission line, lumped reactances, and an ideal transformer. Therefore the equivalent circuits of these nonuniform transmission lines may be represented as mixed lumped and distributed circuits. Furthermore, we applied Kuroda's identity to mixed lumped and nonuniform-distributed circuits and obtained the equivalent circuits of transformed nonuniform transmission lines whose characteristic impedance distributions are represented as rational functions of the distance χ. Therefore, a class of nonuniform transmission lines can be expressed as mixed lumped and distributed circuits, and the network functions of these nonuniform transmission lines can be easily obtained from the equivalent circuits. Finally, we applied these nonuniform transmission lines to an impedance transformer and showed that the characteristics of the nonuniform transmission line has broader band characteristic than the exponential transmission line.

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