Finally, the transform of $R_0(\sigma)$ of (70) is

$$\int_{-\infty}^{\infty} R_0(\sigma) \exp(j\tau\sigma z) d\left(\frac{\sigma z}{2\pi}\right)$$

$$= \exp(-S_0 z) \cdot \exp(\Delta\alpha z) \times [\text{eq. (168) with } \tau \to \tau + \tfrac{1}{2}] \quad (169)$$

yielding (108) and (109).

References

[1] S. E. Miller, "Coupled wave theory and waveguide applications," *Bell Syst. Tech. J.*, vol. 33, pp. 661–720, May 1954.
[2] S. A. Schelkunoff, "Conversion of Maxwell's equations into generalized telegraphist's equations," *Bell Syst. Tech. J.*, vol. 34, pp. 995–1043, Sept. 1955.
[3] W. H. Louisell, *Coupled Mode and Parametric Electronics*. New York: Wiley, 1960.
[4] S. E. Miller, "Waveguide as a communication medium," *Bell Syst. Tech. J.*, vol. 33, pp. 1209–1266, Nov. 1954.
[5] H. E. Rowe and W. D. Warters, "Transmission in mulitmode waveguide with random imperfections," *Bell Syst. Tech. J.*, vol. 41, pp. 1031–1170, May 1962.
[6] A. L. Jones, "Coupling of optical fibers and scattering in fiber," *J. Opt. Soc. Am.*, vol. 55, pp. 261–271, Mar. 1965.
[7] D. Marcuse, "Mode conversion caused by surface imperfections of a dielectric slab waveguide," *Bell Syst. Tech. J.*, vol. 48, pp. 3187–3215, Dec. 1969.
[8] H. E. Rowe, "Approximate solutions for the coupled line equations," *Bell Syst. Tech. J.*, vol. 41, pp. 1011–1029, May 1962.
[9] ——, "Imperfections in active transmission lines," *Bell Syst. Tech. J.*, vol. 43, pt. 1, pp. 261–292, Jan. 1964,
[10] ——, "Propagation in one-dimensional random media," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 73–80, Jan. 1971.
[11] D. T. Young, "Model for relating coupled power equations to coupled amplitude equations," *Bell Syst. Tech. J.*, vol. 42, pp. 2761–2764, Nov. 1963.
[12] H. E. Rowe, *Signals and Noise in Communication Systems*. New York: Van Nostrand, 1965, ch. II, secs. 1.5, 2.8, and 1.2.
[13] D. T. Young, "Effect of differential loss on approximate solutions to the coupled line equations," *Bell Syst. Tech. J.*, vol. 42, pp. 2787–2793, Nov. 1963.
[14] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*. New York: McGraw-Hill, 1955, p. 40, prob. 7.
[15] A. Papoulis, *The Fourier Integral and its Applications*. New York: McGraw-Hill, 1962, ch. 9.
[16] R. Bellman, *Introduction to Matrix Analysis*. New York: McGraw-Hill, 1960, chs. 12 and 15.
[17] P. A. Bello and B. D. Nelin, "The effect of frequency selective fading on intermodulation distortion and subcarrier phase stability in frequency modulation systems," *IEEE Trans. Commun. Syst.*, vol. CS-12, pp. 87–101, Mar. 1964.
[18] G. A. Campbell and R. M. Foster, *Fourier Integrals for Practical Applications*. New York: Van Nostrand, 1948, p. 113, no. 872.2.
[19] S. D. Personick, "Time dispersion in dielectric waveguides," *Bell Syst. Tech. J.*, vol. 50, pp. 843–859, Mar. 1971.
[20] D. T. Young and H. E. Rowe, "Optimum coupling for random guides with frequency-dependent coupling," this issue, pp. 365–372.

Optimum Coupling for Random Guides with Frequency-Dependent Coupling

D. T. YOUNG, MEMBER, IEEE, AND HARRISON E. ROWE, FELLOW, IEEE

***Abstract*—We obtain exactly the covariance of the signal–signal and signal–spurious mode transfer functions of the coupled line equations with two forward-traveling modes, white random coupling with statistically independent successive values (e.g., white Gaussian or Poisson coupling), and a coupling coefficient that varies with the frequency of the signals on the line. No perturbation or other approximations are made in this work. Time-domain statistics for the corresponding impulse responses are obtained for moderate fractional bandwidths.**

These results are extensions of a similar treatment for frequency-independent coupling coefficients, given in a companion paper. If the coupling were independent of frequency, the signal distortion would ultimately decrease as the coupling increased, approaching zero as the coupling approached infinity. The frequency dependence of the coupling coefficient prevents the distortion from approaching zero; the optimum coupling, which achieves minimum signal distortion, is independent of guide length.

Millimeter waveguides and optical fibers with random straightness deviations have coupling coefficients inversely proportional to the frequency. The above results yield the optimum random straightness deviation for such a guide.

More forward modes can be treated in a straightforward way by more complicated calculations.

Manuscript received May 6, 1971; revised July 23, 1971.
D. T. Young was with Bell Telephone Laboratories, Inc., Crawford Hill Laboratory, Holmdel, N. J. 07733.
H. E. Rowe is with Bell Telephone Laboratories, Inc., Crawford Hill Laboratory, Holmdel, N. J. 07733.

I. Introduction

We study exactly the coupled line equations for signal and spurious modes (0 and 1) traveling in the forward direction [1]:

$$I_0'(z) = -\Gamma_0 I_0(z) + jc(z) I_1(z)$$
$$I_1'(z) = jc(z) I_0(z) - \Gamma_1 I_1(z) \quad (1)$$

subject to the initial conditions

$$I_0(0) = 1 \qquad I_1(0) = 0 \quad (2)$$

with coupling coefficient $c(z)$ proportional to a random geometric imperfection $d(z)$

$$c(z) = C \cdot d(z). \tag{3}$$

$d(z)$ is assumed stationary with symmetric probability density; $d(z_1)$ and $d(z_2)$ are assumed statistically independent for $z_1 \neq z_2$. $d(z)$ is consequently white, with (two-sided) spectral density denoted by D_0:

$$\langle d(z+\zeta)d(z)\rangle = D_0 \cdot \delta(\zeta). \tag{4}$$

A special case of interest is white Gaussian $d(z)$ with zero mean. The loss and phase constants are

$$\Gamma_0 \equiv \alpha_0 + j\beta_0 \qquad \Gamma_1 \equiv \alpha_1 + j\beta_1 \tag{5}$$

and the differential loss and phase are

$$\begin{aligned} \Delta\Gamma &\equiv \Gamma_0 - \Gamma_1 \equiv \Delta\alpha + j\Delta\beta \\ \Delta\alpha &= \alpha_0 - \alpha_1 \leq 0 \\ \Delta\beta &= \beta_0 - \beta_1 \end{aligned} \tag{6}$$

as before. Time dependence $\exp(j2\pi ft)$ is assumed. $\Delta\alpha$ must be an even function of the frequency f, $\Delta\beta$ and C odd. The application of this model to real transmission systems, such as millimeter waveguides [4] and optical fibers [5], and the physical interpretation of the solutions to these equations are discussed in [1].

We define [1]

$$\begin{aligned} G_0(z) &\equiv \exp(+\Gamma_0 z) \cdot I_0(z) \\ \boldsymbol{G}_1(z) &\equiv \exp(+\Gamma_0 z) \cdot I_1(z) \end{aligned} \tag{7}$$

as normalized signal–signal and signal–spurious mode transfer functions, respectively. G_0 and $\boldsymbol{G}_1$ are functions of $\Delta\beta$, C, and $\Delta\alpha$. We subsequently neglect the frequency dependence of $\Delta\alpha$. Since $\Delta\beta$ and C are odd functions of f, C is an odd function of $\Delta\beta$. Consequently, we write G_0 and $\boldsymbol{G}_1$ as functions of $\Delta\beta$, with $\Delta\alpha$ as a fixed parameter not indicated by the functional notation. With this convention, the covariances of G_0 and $\boldsymbol{G}_1$ are defined as [1]

$$\begin{aligned} \boldsymbol{R}_0(\sigma) &\equiv \langle G_0(\Delta\beta+\sigma)G_0^*(\Delta\beta)\rangle \\ \boldsymbol{R}_1(\sigma) &\equiv \langle \boldsymbol{G}_1(\Delta\beta+\sigma)\boldsymbol{G}_1^*(\Delta\beta)\rangle \end{aligned} \tag{8}$$

where the boldface notation for the $\boldsymbol{R}$'s indicates that C of (3) above varies with frequency. We distinguish the special case of frequency-independent coupling, i.e., (14) of [1]

$$\begin{aligned} C &= C_0 \cdot \operatorname{sgn} f \\ &= \begin{cases} C_0, & f > 0, \\ -C_0, & f < 0, \end{cases} \quad C_0 \text{ a positive constant} \end{aligned} \tag{9}$$[1]

by denoting the covariances for this case with (nonboldface) R's, as in (57) and (60) of [1]. This convention is adopted because the $\boldsymbol{R}$'s are conveniently found in terms of the solutions for the R's of Section IX of [1].

[1] When $\Delta\beta$ is the independent variable, it may be substituted for f in (9).

II. Exact Transfer-Function Statistics

The analysis of Sections VI, VII, and VIII of [1] remain equally valid for frequency-dependent coupling coefficients—i.e., C is a general odd function of f or $\Delta\beta$, rather than being given by (9) above—with the substitution $S_0 \to S$ throughout:

$$S = C^2 D_0 \tag{10}$$

in place of the second relation of (51) of [1]. Consequently, the expected responses are from (56) of [1]:

$$\begin{aligned} \langle I_0(z)\rangle &= \exp(-\Gamma_0 z)\exp[-(S/2)z] \qquad & \langle I_1(z)\rangle &= 0 \\ \langle G_0(z)\rangle &= \exp[-(S/2)z] & \langle \boldsymbol{G}_1(z)\rangle &= 0. \end{aligned} \tag{11}$$

For second-order transfer-function statistics, the analysis of Section IX of [1] up to and including (65) of [1] remains valid with the above changes, and the notational modification discussed above

$$R \to \boldsymbol{R} \tag{12}$$

to indicate that C is now a general odd function of $\Delta\beta$. Equation (66) of [1] is modified by

$$\begin{aligned} \langle \cos^2 c_i\rangle &\to \langle \cos C_\sigma d_i \cdot \cos C d_i\rangle \\ \pm\langle \sin^2 c_i\rangle &\to \langle \sin C_\sigma d_i \cdot \sin C d_i\rangle \end{aligned} \tag{13}$$

where

$$C_\sigma \equiv C(\Delta\beta + \sigma) \qquad C \equiv C(\Delta\beta). \tag{14}$$

Equation (67) of [1] is replaced by

$$\begin{aligned} \langle \cos C_\sigma d_i \cdot \cos C d_i\rangle &= 1 - \frac{C_\sigma^2 + C^2}{2}\langle d_i^2\rangle + \cdots \\ &= 1 - \frac{C_\sigma^2 + C^2}{2} D_0 \Delta z + \cdots \\ \langle \sin C_\sigma d_i \cdot \sin C d_i\rangle &= C_\sigma C \langle d_i^2\rangle + \cdots \\ &= C_\sigma C D_0 \Delta z + \cdots \end{aligned} \tag{15}$$

to be substituted into (13). Let us define

$$S_\sigma \equiv C_\sigma^2 D_0 \qquad S \equiv C^2 D_0 \tag{16}$$

as the coupling coefficient spectral densities corresponding to the two values of differential phase in the definitions for covariance (8). Let us further define their arithmetic and geometric means as

$$\begin{aligned} S_a &\equiv \frac{S_\sigma + S}{2} = \frac{C_\sigma^2 + C^2}{2} D_0 \\ S_g &\equiv \sqrt{S_\sigma S} = |C_\sigma C| D_0. \end{aligned} \tag{17}$$

Then replacing (68) of [1], we have for the covariances of (8):

$$\begin{aligned} \boldsymbol{R}_0' &= -S_a \boldsymbol{R}_0 \pm S_g \boldsymbol{R}_1 \\ \boldsymbol{R}_1' &= \pm S_g \boldsymbol{R}_0 + (2\Delta\alpha + j\sigma - S_a)\boldsymbol{R}_1 \\ \boldsymbol{R}_0(\sigma)|_{z=0} &= 1 \qquad \boldsymbol{R}_1(\sigma)|_{z=0} = 0. \end{aligned} \tag{18}$$

The sign of $\pm S_g$ is the same as the sign of $C_\sigma C$, i.e., the upper (+) sign applies in the usual case where $\Delta\beta+\sigma$ and $\Delta\beta$ have the same sign (e.g., the two frequencies at which the covariance applies are both positive). Under the transformation

$$\begin{aligned} \boldsymbol{R}_0 &= \exp\left[-(S_a - S_g)z\right]\cdot R_0 \\ \boldsymbol{R}_1 &= \exp\left[-(S_a - S_g)z\right]\cdot R_1 \end{aligned} \tag{19}$$

(18) becomes

$$\begin{aligned} R_0' &= -S_g R_0 \pm S_g R_1 \\ R_1' &= \pm S_g R_0 + (2\Delta\alpha + j\sigma - S_g) R_1 \end{aligned} \tag{20}$$

which is identical to (68) of [1], for frequency-independent coupling, with the substitution $S_0 \to S_g$.

Consequently, the covariances for the normalized signal–signal and signal–spurious mode transfer functions (8) are (from (19) above and (70) of [1])

$$\begin{aligned} \boldsymbol{R}_0(\sigma) = \exp\left[-(S_a - S_g)z\right]\cdot \exp\left[-S_g z(1-\Sigma)\right] \\ \cdot\left[\cosh\left(S_g z\sqrt{1+\Sigma^2}\right) - \Sigma\,\frac{\sinh\left(S_g z\sqrt{1+\Sigma^2}\right)}{\sqrt{1+\Sigma^2}}\right] \end{aligned} \tag{21}$$

$$\begin{aligned} \boldsymbol{R}_1(\sigma) = \pm \exp\left[-(S_a - S_g)z\right]\cdot \exp\left[-S_g z(1-\Sigma)\right] \\ \cdot\frac{\sinh\left(S_g z\sqrt{1+\Sigma^2}\right)}{\sqrt{1+\Sigma^2}} \end{aligned} \tag{22}$$

where

$$\Sigma \equiv \frac{\Delta\alpha + j\sigma/2}{S_g}\,. \tag{23}$$

The coupling coefficient C is a general odd function of $\Delta\beta$ (or of frequency f); $\Delta\alpha$ is constant, independent of $\Delta\beta$. S_a and S_g are defined by (17), S_σ and S by (16) [and (10)], C_σ and C by (14), and D_0 by (4).

Equations (21)–(23) give exact solutions for the second-order transfer-function statistics of the coupled line equations [(1) and (2)] with a white Gaussian geometric imperfection[2] [(3) and (4)] and a general frequency-dependent coupling coefficient, without approximations of any kind. For a coupling coefficient independent of frequency (9), $S = S_\sigma = S_a = S_g \to S_0$ and these results become identical to those of (70) of [1]. The upper (+) sign in (22) for $\boldsymbol{R}_1(\sigma)$ corresponds to the usual case in which the covariance is evaluated at two positive (or two negative) frequencies. We consider only the signal–signal response in the remainder of this paper, and so are interested only in $\boldsymbol{R}_0(\sigma)$ of (21), which contains no such sign ambiguity.

$\boldsymbol{R}_0(\sigma)$ of (21) is a function not only of σ, as indicated by the notation, but also of $\Delta\beta$, since by (14), (16), and (17), S_a and S_g depend on both $\Delta\beta$ and σ. Consequently, the signal–signal transfer function G_0 is *not* wide-sense stationary, except in the special case of (9), frequency-independent coupling [1]. Time-domain statistics are simpler in the wide-sense stationary case [1]. We show in Section III that for moderately narrow fractional bandwidths the $\Delta\beta$ dependence of (21) is small, G_0 is approximately wide-sense stationary, and consequently the second-order time-domain statistics are readily found in terms of the prior results of [1].

[2] See [1, footnote 8].

III. Narrow-Band Approximation

Restricting our attention to moderately narrow fractional bandwidth, C will vary only slightly and linearly with $\Delta\beta$ over the range of interest. Denote the band center by the subscript 0:

$$C_0 \equiv C(\Delta\beta_0) \qquad S_0 = C_0^2 D_0. \tag{24}$$

Then within a modest band

$$C(\Delta\beta) \approx C_0 + C'\cdot(\Delta\beta - \Delta\beta_0) \tag{25}$$

where

$$C' \equiv \left.\frac{dC}{d\Delta\beta}\right|_{\Delta\beta_0} \tag{26}$$

is the derivative of the coupling coefficient. Assume a narrow enough band so that

$$\left|\frac{C'}{C_0}\cdot(\Delta\beta - \Delta\beta_0)\right| \ll 1. \tag{27}$$

From (14), (16), (17), and (24)–(27),

$$S_a - S_g = \left(\frac{C'}{C_0}\right)^2 \frac{S_0}{2}\,\sigma^2 \tag{28}$$

$$\begin{aligned} S_g &\approx S_0\left[1 + \frac{C'}{C_0}(\sigma + 2\Delta\beta - 2\Delta\beta_0)\right] \\ &\approx S_0. \end{aligned} \tag{29}$$

We substitute these relations into (21). Equation (28), substituted into the first exponent of (21), introduces only a σ (and not $\Delta\beta$) dependence. However, S_g, appearing in the second exponent and the hyperbolic arguments of (21), is strictly a function of both σ and $\Delta\beta$; use of the final approximation of (29), regarding S_g as approximately constant by the narrow-band assumption of (27), is required to yield an $\boldsymbol{R}_0(\sigma)$ approximately independent of $\Delta\beta$. We thus obtain

$$\boldsymbol{R}_0(\sigma) \approx \exp\left[-\left(\frac{C'}{C_0}\right)^2 \frac{S_0 z}{2}\,\sigma^2\right]\cdot R_0(\sigma) \tag{30}$$

subject to the restriction of (27). $\boldsymbol{R}_0$ and R_0 are, respectively, the covariances for frequency-dependent and frequency-independent [$C' = 0$, (9)] coupling. R_0 is given by (21) and (23) with $S_a = S_g \to S_0$, or by (70) of [1].

Thus the narrow-band assumption of (27) renders G_0 approximately wide-sense stationary; the covariance $\boldsymbol{R}_0$ for frequency-dependent coupling is approximately the product of a Gaussian function and the covariance R_0 for frequency-independent coupling.

IV. Time-Domain Statistics

We summarize briefly some results of [1]. The normalized signal–signal impulse response is defined as

$$g(\tau) \equiv \int_{-\infty}^{\infty} G_0(\Delta\beta) \exp(-j\tau\Delta\beta z) d\left(\frac{\Delta\beta z}{2\pi}\right). \tag{31}$$

Both g and G_0 are also functions of $\Delta\alpha$, a constant parameter. In [1], C was a constant parameter; in contrast, here C is an arbitrary (odd) function of $\Delta\beta$. Define

$$\boldsymbol{P}(\tau) \equiv \int_{-\infty}^{\infty} \boldsymbol{R}_0(\sigma) \exp(j\tau\sigma z) d\left(\frac{\sigma z}{2\pi}\right) \tag{32}$$

as the spectral density of the (wide-sense stationary) signal–signal transfer function G_0 with covariance $\boldsymbol{R}_0$.

$\Delta\beta$ is an arbitrary odd function of frequency f. The transfer function of independent variable f is $\mathcal{G}(f)$:

$$\mathcal{G}(f) \equiv G_0(\Delta\beta). \tag{33}$$

The impulse response in the time domain t is

$$\mathscr{g}(t) = \int_{-\infty}^{\infty} \mathcal{G}(f) \exp(j2\pi f t) df. \tag{34}$$

Consider the dispersionless case, in which the propagation constants are strictly proportional to frequency, i.e., the mode velocities are strictly constant and, consequently, phase and group velocities are equal, being denoted by v_0 and v_1 for signal and spurious modes, respectively. Then

$$\Delta\beta = -\left(\frac{1}{v_1} - \frac{1}{v_0}\right) \cdot 2\pi f. \tag{35}$$

We assume for convenience that the signal mode is faster:

$$v_0 > v_1. \tag{36}$$

Let

$$T \equiv z\left(\frac{1}{v_1} - \frac{1}{v_0}\right) \tag{37}$$

be the delay between signal and spurious modes for a length z of transmission line. Then

$$\mathscr{g}(t) = \frac{1}{T} g\left(\frac{t}{T}\right) \tag{38}$$

relates normalized and actual impulse responses. τ of (31) is normalized time.

Let the transfer function be wide-sense stationary with covariance

$$\mathcal{R}(\nu) \equiv \langle \mathcal{G}(f+\nu)\mathcal{G}^*(f) \rangle \tag{39}$$

independent of f. Define

$$\mathcal{P}(t) \equiv \int_{-\infty}^{\infty} \mathcal{R}(\nu) \exp(-j2\pi t\nu) d\nu \tag{40}$$

as the spectral density of $\mathcal{G}(f)$. Cascade an ideal bandpass filter of bandwidth $2B$ with $\mathcal{G}(f)$. Then the expected value of the square of the envelope of the resulting impulse response is equal to $8B \cdot \mathcal{P}(-t)$ for large B [1]. Consequently, $\mathcal{P}(-t)$ is the normalized *expected squared envelope of the impulse response*, abbreviated as *pulse response*. In the dispersionless case (35)–(37) give

$$\mathcal{P}(t) = \frac{1}{T} \boldsymbol{P}\left(\frac{t}{T}\right) \tag{41}$$

[compare (38)]. $\boldsymbol{P}(-\tau)$ is consequently the normalized *pulse response*.

The modes in practical media may exhibit dispersion, but may have essentially constant group velocities over a limited band of interest. The above results of this section then apply with v_0 and v_1 replaced by the group velocities v_{g0} and v_{g1}, and (38) replaced by

$$\mathscr{g}_+(t) = \exp(j2\pi t \cdot \text{constant}) \cdot \frac{1}{T} g_+\left(\frac{t}{T}\right)$$

where $_+$ indicates the positive frequency content [2]; since the envelope of $\mathscr{g}$ is unaffected by the exponential factor, nothing else is changed.

Equations (30), (32), and the convolution theorem yield

$$\boldsymbol{P}(\tau) \approx \frac{C_0}{C'}\sqrt{\frac{z}{2\pi S_0}} \exp\left[-\left(\frac{C_0}{C'}\right)^2 \frac{z}{2S_0}\tau^2\right] * P(\tau) \tag{42}$$

subject to the narrow-band restriction of (27), where $\boldsymbol{P}(\tau)$ applies to the present case of frequency-dependent coupling, $P(\tau)$ to the prior results for frequency-independent coupling [1]. From (109) of [1]

$$P(\tau) = \exp(-S_0 z) \cdot \delta(\tau) + P^{ac}(\tau)$$

$$P^{ac}(\tau) = \begin{cases} S_0 z \cdot \exp(-S_0 z) \exp(-2\Delta\alpha z\tau) \sqrt{\dfrac{1+\tau}{-\tau}} \\ \quad \cdot I_1(2S_0 z\sqrt{-\tau(1+\tau)}), & -1 < \tau < 0 \\ 0, & \text{otherwise} \end{cases} \tag{43}$$

where I_1 represents a modified Bessel function of first order.

Equation (42) states that within the narrow-band approximation the frequency dependence of the coupling coefficient modifies the *pulse response* for frequency-independent coupling by convolving it with a Gaussian function whose width is proportional to the frequency dependence. For frequency-independent coupling (9), $C'=0$ and (42) yields $\boldsymbol{P}(\tau) = P(\tau)$, as it must for consistency. Recall from (109) of [1] that the *pulse response* $P(-\tau)$ for frequency-independent coupling is causal and time limited. The approximation to $\boldsymbol{P}(-\tau)$ of (42) for frequency-dependent coupling does not strictly possess these properties because of the narrow-band approximation, although the departures may not be practically significant. Correspondingly, other results of [1] for frequency-independent coupling no longer strictly apply to the present approximation.

V. Large Coupling, $S_0 z \gg 1$

We consider the single special case of large coupling or a long line, $S_0 z \gg 1$. Moreover, initially assume zero differential loss, $\Delta\alpha = 0$ [i.e., the signal- and spurious-mode heat losses are identical (6)]. The simplest approximations of [1] for the signal–signal transfer-function covariance and pulse response under these conditions are given in (132) and (133) of [1]:

$$R_0(\sigma) \approx \tfrac{1}{2} \exp[j(z/2)\sigma] \exp[-(z/8S_0)\sigma^2]$$

$$\begin{aligned} P(-\tau) &\approx P^{ac}(-\tau) \\ &\approx \sqrt{\frac{S_0 z}{2\pi}} \exp[-2S_0 z(\tau - \tfrac{1}{2})^2], \qquad 0 < \tau < 1 \\ &\qquad S_0 z \gg 1, \qquad \Delta\alpha = 0. \end{aligned} \tag{44}$$

From (30), (32), and (42) we have for frequency-dependent coupling

$$\boldsymbol{R}_0(\sigma) \approx \tfrac{1}{2} \exp[j(z/2)\sigma] \cdot \exp\left\{-\left[1 + \left(2S_0 \frac{C'}{C_0}\right)^2\right] \frac{z}{8S_0} \sigma^2\right\}$$

$$\begin{aligned} \boldsymbol{P}(-\tau) \approx & \sqrt{\frac{S_0 z}{2\pi\left[1 + \left(2S_0 \dfrac{C'}{C_0}\right)^2\right]}} \\ & \cdot \exp\left\{-\frac{2S_0 z(\tau - \frac{1}{2})^2}{1 + \left(2S_0 \dfrac{C'}{C_0}\right)^2}\right\}, \qquad 0 < \tau < 1 \\ & S_0 z \gg 1, \qquad \Delta\alpha = 0 \end{aligned} \tag{45}$$

subject to the narrow-band approximation of (27).

$\boldsymbol{P}(-\tau)$ of (45) represents a Gaussian pulse centered on $\tau = \frac{1}{2}$, of width proportional to

$$\Delta\tau = \sqrt{\frac{2}{S_0 z}\left[1 + 2S_0\left(\frac{C'}{C_0}\right)^2\right]}, \qquad S_0 z \gg 1, \qquad \Delta\alpha = 0. \tag{46}$$

For the dispersionless case [see (35)], (41) and (45) give the *pulse response* $\mathcal{P}(t)$ in the actual time domain as approximately Gaussian, centered around $t = (T/2)$, of width

$$\Delta t = T\sqrt{\frac{2}{S_0 z}\left[1 + 2S_0\left(\frac{C'}{C_0}\right)^2\right]}, \qquad S_0 z \gg 1, \qquad \Delta\alpha = 0 \tag{47}$$

T being defined by (37) in the dispersionless case or with the group velocities v_{g0}, v_{g1} substituted for v_0, v_1 in (37) in the more general case, as the delay between signal and spurious modes for a length z of line. The pulse length Δt is a good estimate for the signal–signal impulse-response duration, as shown following (136) in [1].

Proper choice of coupling spectral density S_0 [see (24)] will minimize Δt; denote the minimum value by $\Delta t_{\min}$, the corresponding value of S_0 by S_0^{opt}. Then

$$\frac{\Delta t_{\min}}{T} = 2\sqrt{\frac{2}{z}\left|\frac{C'}{C_0}\right|}, \qquad S_0 z \gg 1, \qquad \Delta\alpha = 0 \tag{48}$$

$$S_0^{\text{opt}} = \frac{1}{2\left|\dfrac{C'}{C_0}\right|}, \qquad S_0 z \gg 1, \qquad \Delta\alpha = 0. \tag{49}$$

We have so far assumed zero differential loss $\Delta\alpha = 0$; the signal loss is [see (5)–(7)] $\frac{1}{2}\exp(-2\alpha_0 z)$. Equations (141)–(144) of [1] show the distortion is unchanged for $\Delta\alpha < 0$ as long as

$$\frac{|\Delta\alpha|}{S_0} \ll 1 \tag{50}$$

the location of the pulse moving from $\tau = (t/T) = \frac{1}{2}$ toward $\tau = t = 0$ as $|\Delta\alpha|$ increases. Thus (46)–(49) remain true if (50) is satisfied. There will be an additional loss $\exp(-|\Delta\alpha| z)$, and, consequently, the signal loss becomes $\frac{1}{2}\exp[-(\alpha_0 + \alpha_1)z]$; i.e., the signal shares the heat losses of the two modes equally [1].

Other cases not discussed here—$S_0 z \lesssim 1$, $|\Delta\alpha| \gtrsim S_0$—are similarly treated using the results of [1]. The present results are of practical interest, and suffice to illustrate the method.

VI. Frequency-Independent Coupling

For frequency-independent coupling (9), $C' = 0$ [see (25)], and the present results become identical to those of [1]. Thus (48) and (49) become $\Delta t_{\min} = 0$ and $S_0^{\text{opt}} = \infty$; for frequency-independent coupling, the duration of the *pulse response* decreases monotonically to 0 as the coupling $S_0 \to \infty$. Setting $C' = 0$ in (47), and using (50):

$$\frac{\Delta t}{T} = \sqrt{\frac{2}{S_0 z}}, \qquad C' = 0, \qquad S_0 z \gg 1, \qquad |\Delta\alpha| \ll S_0. \tag{51}$$

Using (37), (51) may be written alternatively as

$$\Delta t = \left(\frac{1}{v_1} - \frac{1}{v_0}\right)\sqrt{\frac{2z}{S_0}}, \qquad C' = 0, S_0 z \gg 1, \qquad |\Delta\alpha| \ll S_0. \tag{52}$$

where we again substitute group velocities v_{g0}, v_{g1} for v_0, v_1 in the general case.

These results have been obtained in Section XI of [1]. The pulse length is directly proportional to the square root of line length, and inversely proportional to the square root of the coupling coefficient spectral density. Remarkably, the larger the (random) coupling and geometric imperfection, the smaller the signal distortion. This behavior was first discovered by Personick [3] using a different model and analysis. Other cases lying outside the parameter ranges of (51) and (52) are readily treated from (42) and the general results of [1] (i.e., approximations for $P(\tau)$ of (43) for various regions).

Including the frequency dependence of the coupling coefficient prevents the signal distortion from approaching zero. A case of practical importance—white straightness deviation—is treated in Section VII.

VII. Random Straightness Deviation

We consider millimeter metallic waveguides [4] and optical fibers [5] far from cutoff, with white straightness deviation with independent successive values, e.g., white Gaussian straightness deviation. The coupling coefficients are approximately inversely proportional to frequency:

$$C \approx \frac{K_1}{f} \tag{53}$$

where K_1 is a constant. The differential propagation constant may be written (Appendix I of [1]):

$$\Delta\beta \approx \frac{K_2}{f} \tag{54}$$

where K_2 is another constant. The delay between signal and spurious modes is

$$T \equiv z\left(\frac{1}{v_{g1}} - \frac{1}{v_{g0}}\right) = -\frac{z}{2\pi}\frac{d}{df}\Delta\beta \approx \frac{K_2 z}{2\pi f^2} \tag{55}$$

v_g being the group velocities. Making narrow-band approximations, and denoting quantities at the band center by the subscript or superscript 0:

$$T_0 \equiv z\left(\frac{1}{v_{g1}{}^0} - \frac{1}{v_{g0}{}^0}\right) \approx \frac{K_2 z}{2\pi f_0{}^2} \tag{56}$$

$$\Delta\beta \approx \Delta\beta_0 - \frac{2\pi T_0}{z}(f - f_0) \tag{57}$$

where from (54) and (56)

$$\Delta\beta_0 \approx \frac{K_2}{f_0} \approx \frac{2\pi f_0 T_0}{z}. \tag{58}$$

From (53) and (54)

$$C = \frac{K_1}{K_2}\cdot\Delta\beta. \tag{59}$$

Then from (24)–(26) and (58)

$$\frac{C'}{C_0} = \frac{1}{\Delta\beta_0} = \frac{z}{2\pi f_0 T_0} \tag{60}$$

where T_0 is the midband delay between signal and spurious modes for a line of length z, f_0 is the midband frequency.

Substitution of (60) into (48)–(50) yields the minimum pulse length $\Delta t_{\min}$ as

$$\frac{\Delta t_{\min}}{T_0} = \frac{2}{\sqrt{\pi f_0 T_0}} = 2\sqrt{\frac{2}{K_2}}\sqrt{\frac{f_0}{z}}$$

$$\Delta t_{\min} = 2\sqrt{\frac{T_0}{\pi f_0}} = \frac{\sqrt{2K_2}}{\pi}\frac{1}{f_0}\sqrt{\frac{z}{f_0}}$$

$$S_0 z \gg 1, \qquad |\Delta\alpha| \ll S_0 \tag{61}$$

obtained for a coupling spectral density

$$S_0{}^{\text{opt}} = \frac{\pi f_0 T_0}{z} = \frac{K_2}{2f_0}. \tag{62}$$

The useful RF bandwidth W is proportional to the reciprocal of the duration of the impulse response, approximately Δt of (47). Let us define the constant of proportionality by

$$W \equiv \frac{4}{\pi}\cdot\frac{1}{\Delta t} \tag{63}$$

for later convenience. Then the maximum useful bandwidth $W_{\max}$ corresponds to the minimum pulse length $\Delta t_{\min}$ of (61):

$$\frac{W_{\max}}{f_0} = \frac{2}{\sqrt{\pi f_0 T_0}} = 2\sqrt{\frac{2}{K_2}}\sqrt{\frac{f_0}{z}}. \tag{64}$$

Comparing (64) and (61),

$$\frac{\Delta t_{\min}}{T_0} = \frac{W_{\max}}{f_0}. \tag{65}$$

the maximum fractional bandwidth equals the minimum fractional pulse length, for a coupling coefficient with inverse frequency dependence.

The large coupling restriction $S_0 z \gg 1$ of (61) guarantees via (62) that

$$f_0 T_0 \gg 1 \tag{66}$$

i.e., the delay between signal and spurious modes must be long compared to the carrier period. Thus from (61), (64) and (65),

$$\frac{\Delta t_{\min}}{T_0} = \frac{W_{\max}}{f_0} \ll 1. \tag{67}$$

This is consistent with the restriction of (27), which using (57) and (60) becomes

$$\frac{|f - f_0|}{f_0} \ll 1 \tag{68}$$

i.e., the fractional bandwidth must remain small.

Finally, it is of interest to determine the optimum straightness deviation spectral density. From (10), (24), (53), (58), and (62)

$$D_0{}^{\text{opt}} = \frac{K_2}{2K_1{}^2} f_0 \tag{69}$$

K_1 and K_2 being the constants of (53) and (54). The straightness deviation spectral density required for optimum transmission is proportional to the carrier frequency.

The above results establish the minimum pulse length and maximum useful bandwidth that can be attained in two-mode guide with white straightness deviation with independent successive values, by varying the amount of straightness deviation (and hence coupling). As the coupling S_0z increases from some moderate value much greater than 1 to larger values, the pulse length Δt will initially decrease according to (51), but will eventually reach a minimum value given by (61), at a value of coupling given by (62); correspondingly, the useful bandwidth W will increase and reach the maximum given by (64). The signal transmission can no longer be indefinitely improved by increasing the coupling, as obtained in [1] and the previous section with the coupling-coefficient frequency dependence neglected. The minimum fractional pulse length and the maximum fractional bandwidth are equal, as shown in (65), and both are small compared to 1. The more pulse shortening you can get, the smaller the available fractional bandwidth; the larger the available fractional bandwidth, the less the possible pulse shortening. The minimum pulse length increases and the maximum bandwidth decreases as the square root of line length. If the optimum coupling is too large to be obtained practically, then the simpler results of (51) and (52) (for frequency-independent coupling) may be used, but of course the present results are required to know this. $1/S_0$ represents a characteristic length in which power introduced in one mode becomes approximately equally divided between both modes [1]. The optimum coupling (62) is independent of line length. The optimum geometric imperfection (69) varies linearly with frequency, and is of course also independent of line length.

The results of this section are appropriate to a much wider class of imperfections than straightness deviation; for example, general small deformations of metallic guide (e.g., ellipticity, trifoil, etc.) have coupling coefficients with inverse frequency dependence as assumed in (53) [4]. These results are restricted to the region where the coupling coefficient and differential propagation constant vary approximately inversely with frequency (53) and (54), i.e., far from cutoff.

VIII. Discussion

For the coupled line equations with a white geometric imperfection (and hence white coupling) with independent successive values, e.g., white Gaussian imperfection (and coupling), neglecting the frequency dependence of the coupling, the transmission improves indefinitely as the coupling (and geometric imperfection) increases. However, a frequency-independent coupling coefficient violates causality in these equations [1]. The present work includes the frequency dependence of the coupling coefficient; this factor limits the transmission improvement that can be obtained by increasing the random coupling.

Suppose that more-or-less flat filters are used at both ends of such a random guide, which has an impulse-response duration Δt and corresponding bandwidth W as given above, perhaps optimum or perhaps not. Best use of the available frequency space dictates that the terminal filter bandwidths be not too much greater than W.

It may seem surprising that the frequency dependence of the coupling coefficient can become significant even though the fractional bandwidth is small. The normalized signal–signal transfer function may be written from (153) and (154) of [1] and from (3) of this paper as

$$G_0(z) = 1 + \sum_{n=1}^{\infty} (-1)^n G_{0(n)}(z) \tag{70}$$

$$\begin{aligned} G_{0(n)}(z) \equiv \int_0^z dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{2n}} dx_{2n} \\ \cdot C^{2n} \cdot d(x_1) d(x_2) d(x_3) \cdots d(x_{2n}) \\ \cdot \exp\left[j\Delta\beta(x_1 - x_2 + x_3 - x_4 + \cdots - x_{2n})\right] \end{aligned} \tag{71}$$

where C is a general odd function of $\Delta\beta$, with narrow-band behavior given by (24)–(27). The nth term represents the sum of all signals that have suffered exactly $2n$ transitions between the two modes. The larger the coupling and geometric imperfection $d(x)$, the greater the number of terms n that must be taken. The nth term is proportional to C^{2n}; even a slight frequency dependence of C will eventually become significant if n becomes large enough. While (70) and (71) render the present results physically plausible, it does not appear easy to obtain our present results directly from them.

The general results of the present work, stated in (21)–(23), give the transfer-function covariances without approximation for arbitrary frequency dependence of the coupling coefficient. The transfer-function statistics are in general nonstationary; since stationary statistics are simpler to deal with, approximations were made in subsequent examples to render the transfer function approximately wide-sense stationary. Moreover, we neglected dispersion and assumed large coupling $S_0z \gg 1$, small loss $|\Delta\alpha| \ll S_0$, and coupling coefficient dependence as that far from cutoff. None of these approximations or assumptions are essential. Extension to other regions of coupling and loss is accomplished by convolution (42) with other approximations of [1] to (43). Other restrictions probably can and should be removed. Finally, the dispersion of the signal-mode propagation factor $\exp(-j\beta_0 z)$ removed in the initial normalization of (7) has not been studied here, but is equivalent to cascading a deterministic filter with the present random transfer function.

The differential heat loss $\Delta\alpha = \alpha_0 - \alpha_1$ does not affect the transmission distortion as long as $|\Delta\alpha| \ll S_0$, i.e., heat loss much less than coupling. It does, as noted in Section V, affect the overall signal loss, which is $\frac{1}{2}\exp[-(\alpha_0+\alpha_1)z]$, the signal sharing the heat losses of the two modes for $S_0z \gg 1$ [1].

This work may be extended to more forward spurious modes (but not backward spurious modes), and to systems whose output is a combination of several modes.

We do not know if the present limitations on impulse-response duration and bandwidth are fundamental or whether equalization might improve things. The latter possibility is suggested by the above discussion in connection with (70) and (71); for a given (large) coupling, the significant terms will tend to cluster around a specific value of n, and hence equalization might tend to help, although it could obviously never be perfect.

A single value for the spectral density S_0^{opt} of the coupling coefficient (or D_0^{opt} of the geometric imperfection) yields the shortest impulse response (and widest useful bandwidth) for all line lengths. This result was unforeseen.

References

[1] H. E. Rowe and D. T. Young, "Transmission distortion in multimode random waveguides," this issue, pp. 349–365.
[2] H. E. Rowe, *Signals and Noise in Communication Systems.* New York: Van Nostrand, 1965, ch. II, secs. 1.5 and 2.8.
[3] S. D. Personick, "Time dispersion in dielectric waveguides," *Bell Syst. Tech. J.*, vol. 50, pp. 843–859, Mar. 1971.
[4] H. E. Rowe and W. D. Warters, "Transmission in multimode waveguide with random imperfections," *Bell Syst. Tech. J.*, vol. 41, pp. 1031–1170, May 1962.
[5] D. Marcuse, "Mode conversion caused by surface imperfections of a dielectric slab waveguide," *Bell Syst. Tech. J.*, vol. 48, pp. 3187–3215, Dec. 1969.

Simplified Theory for Post Coupling Gunn Diodes to Waveguide

JOSEPH F. WHITE, MEMBER, IEEE

***Abstract*—There is a constant need for diode circuits employing rectangular waveguide. Coupling of the diode to the guide by using an inductive post is a popular method. The microwave circuit analysis of the equivalent circuit has been explored by complete theoretical analyses in the literature, but the results have been sufficiently difficult to apply that, in practice, recourse is often made to empiric characterization. This paper derives a simplified equivalent circuit based on a small perturbation approximation. The method is verified by experiment and is then used to evaluate a practical Gunn oscillator cavity.**

I. Introduction

COUPLING DIODES to rectangular waveguides is a frequent requirement in the design of microwave oscillators, detectors, and control devices. In effect, a mode transducer is required to couple the diode, which is essentially a lumped element with dimensions small compared to a wavelength, to the dominant TE_{10} mode whose fields are not nearly as concentrated in space as those surrounding the diode.

High Q resonators are very practical in waveguide, and this is advantageous with bulk-effect diodes whose operating frequency must be stabilized. Post coupling

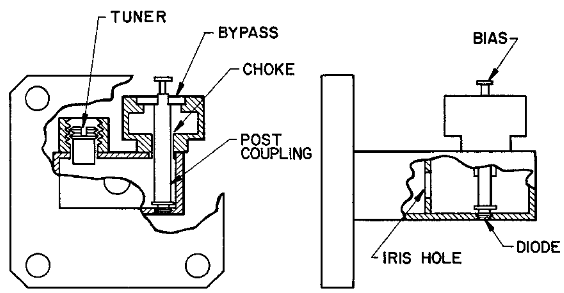

Fig. 1. Oscillator plan view.

the diode to the guide, as shown in Fig. 1, is common. The problem for the designer is how to estimate what load impedance will be experienced by the diode in this network.

The equivalent circuit that we will use is shown in Fig. 2. The reactance X_g associated with the gap will be neglected, and the conditions under which this approximation is valid will be discussed. The normalized[1] impedance $\bar{Z}_{ss'}$ can be calculated using transmission line theory and published [1]–[3] or measured values for the

Manuscript received March 11, 1971; revised October 13, 1971.
The author is with Microwave Associates Incorporated, Burlington, Mass. 01803.

[1] In this paper, a bar over a variable denotes that its value is normalized to Z_g, the waveguide impedance.