If we are interested in these mean values for only a limited frequency range $f_a<f<f_b$, write $T=L/c$ for the time required for a wave to propagate the length L of the line, and assume that the spectral density function $w(f)$ has a uniform value w_0 between f_a and f_b (as it does by even the quantum mechanical form of the Nyquist noise formula up to nearly the highest microwave frequencies in current use):

$$\overline{E} = Tw_0(f_b - f_a) \quad \text{and} \quad \overline{\epsilon^2} = Tw_0^2(f_b - f_a). \tag{29}$$

Eliminating w_0 by substituting the first of these formulas into the second and redesignating T as Δt and f_b-f_a as Δf gives

$$\overline{\epsilon^2} = \frac{\overline{E}^2}{\Delta t \Delta f}. \tag{30}$$

Rice[10,11] obtains for the mean energy dissipated in a one-ohm resistor by a noise current with uniform spectral density w_0 during time T in bandwidth f_b-f_a

$$\overline{E} = Tw_0(f_b - f_a), \tag{31}$$

and for the mean-square energy fluctuation

$$\overline{\sigma_T^2} = w_0^2T(f_b - f_a). \tag{32}$$

By similarly eliminating w_0 between these formulas, this mean-square fluctuation formula can also be converted to the characteristic form $\overline{E}^2/\Delta t\Delta f$.

ACKNOWLEDGMENT

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On the Resolution of a Class of Waveguide Discontinuity Problems by the Use of Singular Integral Equations*

L. LEWIN†

Summary—**It is shown that a considerable number of solutions of rectangular waveguide problems appearing in the literature are all special cases of a general treatment focused around the known solution of a singular integral equation. In terms of this a number of typical results are re-examined. The method is then applied to four new configurations, and the range of application and the limitations are examined.**

I. Introduction

The number of waveguide problems capable of exact solution is limited to a few very simple shapes, even when the common approximations of ideal geometry and infinite wall conductivity are made. A class of problems recently amenable to exact treatment has involved configurations in which the discontinuity has separated the space into two uniform regions, $z<0$ and $z>0$. Examples are the radiation into free space of a semi-infinite length of guide, a bifurcation of the waveguide, and, exceptionally, a diaphragm halfway across the guide. The solutions involve the setting up of an integral equation for the field along the guide axis, or some other equivalent axis, the integral equation taking a different form on either side of the discontinuity. It is then solved by the Wiener-Hopf technique, the waveguide parameters being readily obtainable from the solution.

This method gives a rigorous result for the limited number of configurations to which it can be applied. It is not successful, however, in the majority of those cases in which the discontinuity takes the form of a variation over the cross section of the waveguide, such as, for example, diaphragms, strips, change of guide cross section, etc. Nor is it applicable to configurations in which the propagation medium changes at the discontinuity, *e.g.*, if there is a dielectric or ferrite insert.

For such cases it is more satisfactory to take the field over the cross section as the unknown variable, and a different type of integral equation can be set up for this class of problems. The Wiener-Hopf technique is no longer usuable, but the equation can be solved to various quasi-static degrees of approximation in some particular cases. This has been done by Schwinger and co-authors[1] for waveguide diaphragms, and by Lewin[2,3] for un-

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[1] N. Marcuvitz, "Waveguide Handbook," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., p. 147; 1951.
[2] L. Lewin, "The impedance of unsymmetrical strips in rectangular waveguides," *Proc. IEE*, vol. 99, pt. 4, pp. 168–176, Monograph No. 29; 1952.
[3] L. Lewin, "A ferrite boundary value problem in a rectangular waveguide," *Proc. IEE*, vol. 106, pt. B, pp. 559–563; November, 1959.

symmetrical waveguide strips, and for a ferrite-loaded guide section. Schwinger[4] has also used conformal transformation methods to obtain expressions for the waveguide parameters without obtaining in explicit form the integral equation solution. This last method is very powerful, and its results include those for some of the diaphragm configurations, otherwise obtainable by the direct solution of the quasi-static integral equation.

However, not all such configurations can be catered for in this way; and it appears that a direct solution of the quasi-static integral equation is necessary for unsymmetrical inductive configurations, changes of propagation medium, reactive strips, and others.

It is with this latter class of discontinuities that this paper is concerned. The equations have occurred sporadically in the literature and have been solved by *ad hoc* methods. It is now realized that they are all particular cases of a general treatment which has a wide, albeit limited, field of applicability. The paper outlines first the known examples mentioned above, as particular cases of the general treatment, and finishes with a few new examples and an indication of the types of configuration to which the method should be successful. It does not, of course, displace the earlier treatments; rather it extends the range of problems that can, to the various quasi-static degrees of approximation chosen, be rigorously solved.

II. Inductive Diaphragm

Fig. 1 shows an inductive diaphragm in a rectangular waveguide. A wave in the dominant mode, $E_x = \sin(\pi y/a)e^{-jk'z}$, is incident from $z = -\infty$ and sets up a reflected wave, a transmitted wave, and a train of evanescent modes on both sides of the diaphragm. The field in the aperture, $E(y)$, is as yet unknown, but in terms of it, by a Fourier expansion, the amplitudes of the various modes can be expressed. The continuity of tangential magnetic field over the aperture leads to an equation containing the mode amplitudes, and if these be expressed, as above, in terms of $E(y)$ an integral equation for $E(y)$ results. This equation can be simplified by an integration by parts, and, in terms of the unknown diaphragm susceptance, B, takes the form[5]

$$B \sin(\pi y/a) \int F(\eta) \cos(\pi\eta/a) d\eta = -\sum_{2}^{\infty} (\lambda_G/a)(1-\delta_n) \int F(\eta) \sin(n\pi y/a) \cos(n\pi\eta/a) d\eta. \quad (1)$$

Here $F(\eta) = E'(\eta)$, the variable of integration, η, ranging only over the aperture, which is also the range of y over which (1) has to hold. The quantity $\delta_n = 1-(1-k^2a^2/n^2\pi^2)^{1/2}$ is a small quantity, vanishing for high-mode number, n, and is a measure of the departure of the nth mode from its quasi-static value. As a first approximation it can be neglected. As a second, the first one or two coefficients are retained, with a corresponding higher-order solution resulting. The retention of these higher-order terms does not affect the method of solution, though it complicates it. They will therefore not be considered here, though their inclusion at any point is relatively straightforward.

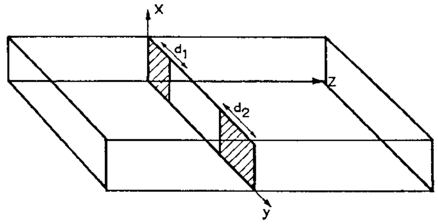

Fig. 1—Inductive diaphragm in rectangular waveguide.

In order to solve (1) with δ_n neglected, we add and subtract the first term of the infinite series, summing it via the known result

$$\sum_{1}^{\infty} \frac{\cos(n\pi y/a)\cos(n\pi\eta/a)}{n} = -\tfrac{1}{2}\log 2\left|\cos(\pi\eta/a) - \cos(\pi y/a)\right|.$$

On differentiation with respect to y this gives

$$\sum_{1}^{\infty} \sin(n\pi y/a)\cos(n\pi\eta/a) = \frac{\tfrac{1}{2}\sin(\pi y/a)}{\cos(\pi\eta/a) - \cos(\pi y/a)}, \quad (2)$$

and substituting into (1) gives

$$(2 + 2aB/\lambda_g) \int F(\eta)\cos(\pi\eta/a) d\eta = -\int \frac{F(\eta) d\eta}{\cos(\pi\eta/a) - \cos(\pi y/a)}, \quad (3)$$

where, as before, the range of both y and η is the diaphragm aperture.

Now the whole of the left-hand side is some constant, C, independent of y. If we take new variables $X = \cos(\pi\eta/a)$, $Y = \cos(\pi y/a)$ and put $F(\eta)d\eta = G(X)dX$ then (3) becomes

$$\int_A^B \frac{G(X)dX}{X - Y} = -C \qquad A < Y < B. \quad (4)$$

Here A and B are the new limits for X corresponding to the aperture limits for η. The inclusion of higher-order terms would add a polynomial in Y to the right-hand side; otherwise the form of (4) is unaltered.

The principal value of the integral in (4) is to be understood. Hence (4) is a singular integral equation, and we can appeal to the theory of these equations for its solution. A convenient reference is Tricomi,[6] where

[4] Marcuvitz, *op. cit.*, p. 156.

[5] L. Lewin, "Advanced Theory of Waveguides," Iliffe and Sons, London, England, p. 47; 1951.

[6] F. G. Tricomi, "Integral Equations," Interscience Publishers, New York, N. Y., pp. 173–188; 1957.

we find the solution of

$$f(X) = \frac{1}{\pi}\int_{-1}^{1} \frac{\phi(Y)}{Y - X} dY \tag{5}$$

is given by

$$\phi(X) = -\frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1 - Y^2}{1 - X^2}} \frac{f(Y)}{Y - X} dY + \frac{K}{\sqrt{1 - X^2}}. \tag{6}$$

K is of the nature of an integration constant. To apply this result to (4) it is necessary to change the range of integration from (A, B) to $(-1, 1)$. This is easily done by taking new variables

$$X' = \frac{2(X - A)}{B - A} - 1 \tag{7}$$

with a similar form for Y' in terms of Y. This is precisely equivalent to Schwinger's transformation[1] $\cos(\pi y/a) = c + s \cos\theta$ which is the "trick" by means of which such equations as (3) have been treated hitherto. Now K in (6) is

$$\frac{1}{\pi}\int_{-1}^{1} \phi(X) dX$$

which, in the case of (4), reduces to

$$\int G(X) dX = \int F(\eta) d\eta = E(\eta).$$

For physical reasons this vanishes at both limits; thus $K = 0$ and the additional term in (6) vanishes. $f(Y)$ is a constant in the present case, and since

$$\int_{-1}^{1} \frac{\sqrt{1 - Y^2} dY}{Y - X} = -\pi X,$$

the solution to (4) follows at once. Corresponding solutions are obtained if the right-hand side of (4) is replaced by a finite polynomial so that higher-order solutions are readily obtained without any *ad hoc* guessing at the necessary forms.

The details of these expressions do not concern us here. The important thing is the realization that (3) can be reduced to a special case of (5) by means of a simple change of variable. It is the constant reappearance of the singular integral equation in various forms that is the key to the extension of the method to a wider range of configurations.

III. The Unsymmetrical Capacitive Strip

The setting up of the integral equation follows a similar route to that of the previous case except that the current on the strip rather than the aperture field is the unknown to be evaluated. (A similar type of expression results when the current rather than the aperture field is used in the diaphragm cases.) The quasi-static integral equation can be put in the form[2]

$$x + C = 2\sum_{1}^{\infty} \int \sin(n\pi x/b) \cos(n\pi\xi/b) I(\xi) d\xi, \tag{8}$$

where C is a constant, $I(\xi)$ is proportional to the unknown current in the strip, and the range of both x and ξ is over the strip.

This equation, on using (2) to effect the summation, and on putting $\cos(\pi x/b) = X$, $\cos(\pi\xi/b) = Y$, $I(\xi)d\xi = F(Y)dY$ becomes

$$\frac{C + \frac{b}{\pi}\cos^{-1} X}{\sqrt{1 - X^2}} = \int \frac{F(Y) dY}{Y - X}. \tag{9}$$

The change of variable of (7) at once reduces this to the form (5) and the solution, apart from the rather awkward integration, follows. Integration constants, which enter in this problem, are determined by setting the tangential electric field zero at the strip edge.

In the original paper[2] (8) was solved through the use of Schwinger's transformation, assuming an infinite Fourier series for $I(\xi)$. This was eventually summed, leading to the same result as the solution to (9). The details, together with the final integrations, are given in the reference.

The point to be noted here is the reappearance of the singular integral equation in (9), albeit in a more complicated form than in (4).

IV. Ferrite-Loaded Waveguide

The arrangement consists of a rectangular waveguide filled with a medium of dielectric constant ϵ for $-\infty < z < 0$, and transversely magnetized ferrite for $0 < z < \infty$. The reason that this arrangement gives rise to anything more involved than simple reflected and transmitted waves is that the ferrite supports a magnetic field distribution which differs, on account of the tensor permeability, from that in the plain guide. Hence an infinite series of modes is needed, on both sides of the boundary, to satisfy continuity conditions. The details are given in Sharpe and Heim's paper.[7] In Lewin[3] the integral equation derived is of the form

$$C = MF(Y) + \frac{jK}{\pi}\int_{-1}^{1} \frac{F(X)}{X - Y} dX, \tag{10}$$

where C, M and K are constant, and $F(X)$ is related to the field at the junction across the ferrite face.

This is a singular integral equation, but of a different sort from (5). In the reference it was solved partly by guesswork; but it is a particular case of Carleman's equation[6]

$$a(x)\phi(x) - \lambda \int_{-1}^{1} \frac{\phi(y)}{y - x} dy = f(x), \tag{11}$$

[7] C. B. Sharpe and D. S. Heim, "A ferrite boundary-value problem in a rectangular waveguide," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 42–46; January, 1958.

with the solution

$$\phi(x) = \frac{a(x)f(x)}{a^2(x) + \lambda^2\pi^2} + \frac{\lambda e^{\tau(x)}}{\sqrt{a^2(x) + \lambda^2\pi^2}} \cdot \left[\int_{-1}^{1} \frac{e^{-\tau(y)}f(y)}{\sqrt{a^2(y) + \lambda^2\pi^2}} \frac{dy}{y - x} + \frac{c}{1 - x}\right], \quad (12)$$

where

$$\tau(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\theta(t)}{t - x} dt, \qquad \theta(t) = \tan^{-1}\frac{\lambda\pi}{a(t)} \cdot$$

All the integrals involved are principal values.

Eq. (12) contains (5) as the special case $a(x) = 0$. Eq. (10) is the case $a(x) = M$, $f(x) = C$: higher-order mode solutions replace C by a simple polynomial, with no change in the character of the solution.

Although solutions have, in the past, been obtained, partially by guesswork, the formulation of the explicit form (11) and its solution (12), with (5) and (6) as a special case, is the central feature around which this paper is written. Some new extensions of existing configurations follow.

V. Ferrite-Loaded Waveguide with Inductive Diaphragm

Although the solution of the ferrite-loaded waveguide of Section IV is quite a formidable task, it is not, in fact, much more difficult to combine it with an inductive diaphragm. The arrangement is shown in Fig. 2, the diaphragm being located at the ferrite face.

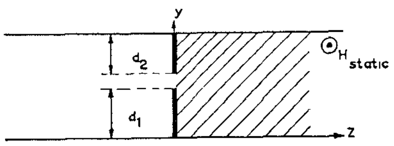

Fig. 2—Inductive diaphragm in ferrite-loaded guide.

If the process of setting up the integral equation is repeated, with the addition of the metallic diaphragm at the boundary, it is seen that the only changes occasioned by the alteration is a reduction of the range of integration, and of the independent variable in the integral equation, to the new aperture. Hence, the new equation is

$$C = MF(y) + j\frac{K}{\pi}\int_{\alpha}^{\beta} \frac{F(x)}{x - y} dx, \quad (13)$$

where $-C = 1 + KX$ and the normalized reactance, X, is

$$jX = \frac{2}{\pi}\int_{\alpha}^{\beta} yF(y)dy. \quad (14)$$

These forms are taken from Lewin.[3] Moreover, $F(y)$ must satisfy

$$\int_{\alpha}^{\beta} F(y)dy = 0.$$

If d_1 and d_2 are the diaphragm inserts, then the values of α and β are given by $-\cos(\pi d_2/a)$ and $\cos(\pi d_1/a)$. The transformation (7) gives new limits $(1, -1)$ with

$$\frac{dx}{x - y} = \frac{dx'}{x' - y'} \cdot$$

Hence the equation transforms unaltered into (10), and the only eventual change is that (14) for jX becomes multiplied by the factor $f^2 = \frac{1}{4}(\beta - \alpha)^2$ from the contribution of ydy on changing variables. From (33) of Lewin[3] we accordingly get the equation

$$X = -\frac{4C}{K}p(1 - p)f^2 \text{ instead of } -\frac{4C}{K}p(1 - p),$$

which, together with $-C = 1 + KX$ gives, eventually,

$$X = -\frac{1}{K}\left\{1 + \frac{\pi^2}{f^2L^2 + \pi^2(f^2 - 1)}\right\}, \quad (15)$$

where

$$L = \frac{1}{\pi}\log\frac{K + M}{K - M}$$

(see (35) of Lewin[3]). The factor f can be put in the form $\sin(\pi d/2a)\sin(\pi y_0/a)$ where $d = a - d_1 - d_2$ is the aperture opening and $y_0 = \frac{1}{2}(a + d_1 - d_2)$ is the coordinate of its center. Eq. (15) reduces to (35) of Lewin[3] when $f = 1$ (no diaphragm inserts) and gives $X = 0$, as it must, for $f = 0$ (diaphragm completely across the guide).

VI. Rectangular Waveguide Bifurcation (H-plane)

Fig. 3 shows an H-plane bifurcation of a rectangular waveguide, in which two guides of width a join at $z = 0$ into a single guide of width $2a$.

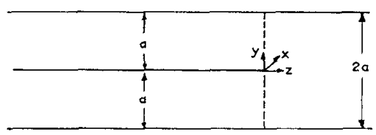

Fig. 3—H-plane bifurcation in rectangular waveguide.

This arrangement in it simplest form is solvable rigorously by the Wiener-Hopf technique.[8] The justification of treating it here by the quasi-static integral equation method is that a number of variants are possible which yield only to the latter method of attack. Thus, a different dielectric material can be used on either side of the junction, or an arrangement of diaphragm or strips can be incorporated there, or both of these can be used together. These variants are foreign to the Wiener-Hopf approach, which, nevertheless, has its own field of applicability, *e.g.*, to bifurcation with unequal guides. It just so happens that the two approaches overlap in the simple arrangement of Fig. 3.

[8] Marcuvitz, *op. cit.*, p. 383.

Two basic modes can be supported by Fig. 3, symmetrical and antisymmetrical. Any method of feeding the two guides from the left can be resolved into a sum of these two. Moreover, since the antisymmetrical mode is just the natural second-order mode in the broad guide, there is no change of field at the junction—the field propagates without reflection.

Hence only the symmetrical mode needs to be considered, and we confine our attention to the upper part of the figure, $0<y<a$.

For $z<0$ we have

$$E_x = (e^{-jk'z} + Re^{jk'z}) \sin (\pi y/a) + \sum_2^\infty R_n e^{\gamma_n z} \sin (n\pi y/ \qquad (16)$$

$$Z_0 H_y = (e^{-jk'z} - Re^{jk'z})(k'/k) \sin (\pi y/a) + j \sum_2^\infty R_n(\gamma_n/k) e^{\gamma_n z} \sin (n\pi y/a), \qquad (17)$$

where $Z_0=120\pi$, $k'=2\pi/\lambda_g$ and $\gamma_n=\sqrt{n^2\pi^2/a^2-k^2} \sim n\pi/a$ for large n. The reflection coefficient R and the mode amplitudes R_n have yet to be determined.

For $z>0$, bearing in mind the symmetrical feeding,

$$E_x = T_1 e^{-jK'z} \cos (\pi y/2a) + \sum_1^\infty T_{2m+1} e^{-\Gamma_{2m+1} z} \cos (\overline{2m+1}\, \pi y/2a) \qquad (18)$$

$$Z_0 H_y = T_1(K'/k) e^{-jK'z} \cos (\pi y/2a) - j \sum_1^\infty T_{2m+1}(\Gamma_{2m+1}/k) e^{-\Gamma_{2m+1} z} \cos (\overline{2m+1}\, \pi y/2a). \qquad (19)$$

Here $K'=2\pi/\Lambda_g$ with $\Lambda_g=\lambda/\sqrt{1-(\lambda/4a)^2}$ and

$$\Gamma_{2m+1} = \sqrt{(2m+1)^2\pi^2/4a^2 - k^2} \sim (2m+1)\pi/2a \text{ for large } m.$$

If $E(\pi y/a)$ is the as yet unknown field in the aperture $(0<y<a)$, then the various mode amplitudes can be expressed in terms of it as follows:

$$1 + R = \frac{2}{a}\int_0^a E(\pi\eta/a) \sin (\pi\eta/a) d\eta = \frac{2}{\pi}\int_0^\pi E'(\theta) \cos \theta d\theta \qquad (20)$$

on putting $\pi\eta/a=\theta$ and integrating by parts. (The integrated part vanishes at both limits because of the vanishing of the tangential electric field at the metal boundaries.) Similarly,

$$R_n = \frac{2}{\pi n}\int_0^\pi E'(\theta) \cos n\theta d\theta \qquad (21)$$

$$T_{2m+1} = \frac{-4}{\pi(2m+1)}\int_0^\pi E'(\theta) \sin (\overline{2m+1}\, \theta/2) d\theta. \qquad (22)$$

These values can be substituted in (17) and (19). We replace γ_n and Γ_{2m+1} by their dominant forms and a remainder term, and write $\pi y/a=\phi$ as a conplementary variable to θ. The summations are affected via (2), there being some simplification of terms. The equation expressing continuity of tangential magnetic field over the aperture is obtained by equating (17) to (19) for $0<y<a$.

$$A \sin \phi + B \cos (\phi/2) = \int_0^\pi \frac{E'(\theta) \cos (\phi/2) d\theta}{2[\sin (\theta/2) - \sin (\phi/2)]} + S, \qquad (23)$$

where

$$A = -\tfrac{1}{2}jk'a(1-R) - \int_0^\pi E'(\theta) \cos \theta d\theta,$$

$$B = (1 - j2aK'/\pi)\int_0^\pi E'(\theta) \sin (\theta/2) d\theta.$$

S is a remainder term giving the effect of the differences of the higher-order mode attenuation constants from their quasi-static values.

$$S = \sum_1^\infty \left(\frac{2a\Gamma_{2m+1}}{\pi(2m+1)} - 1\right)\int_0^\pi E'(\theta) \sin (\overline{2m+1}\, \theta/2) \cdot \cos (\overline{2m+1}\, \phi/2) d\theta - \sum_2^\infty \left(\frac{a\gamma_n}{n\pi} - 1\right)\int_0^\pi E'(\theta) \sin (n\phi) \cos (n\theta) d\theta. \qquad (24)$$

To solve (23) to the quasi-static approximation, neglect S and change variables by putting $\sin (\theta/2)=\frac{1}{2}(1+x)$, $\sin (\phi/2)=\frac{1}{2}(1+y)$. A common factor $\cos (\phi/2)$ can be cancelled in (23) which becones, on putting $E'(\theta)d\theta = F(x)dx$,

$$A(1+y) + B = \int_{-1}^1 \frac{F(x)dx}{x-y}. \qquad (25)$$

This is of the form (5), giving as solution,

$$F(x) = \frac{1}{\pi^2\sqrt{1-x^2}}[C + \pi x(A + B + Ax)]. \qquad (26)$$

In order to determine C, we note that

$$\int_{-1}^1 F(x)dx = \int_0^\pi E'(\theta)d\theta = E(\pi) - E(0) = 0,$$

since E vanishes at the limits. Finally, therefore,

$$F(x) = \frac{1}{\pi\sqrt{1-x^2}}[Ax^2 + (A+B)x - \tfrac{1}{2}A]. \qquad (27)$$

From (20), together with the two equations defining A and B, we get three relations from which A and B can be eliminated. These give the following expression for the normalized quasi-static impedance at the junction,

$$Z = \frac{1+R}{1-R} = j\frac{k'a}{\pi}\,\frac{35\pi + j2K'a}{13\pi + j30K'a}\,. \qquad (28)$$

If only one waveguide is fed from the left we have to add an antisymmetrical mode of unit amplitude to cancel the wave in the other guide. Hence the incident mode amplitude is now 2 so that in this case the *relative* reflection coefficient is $\frac{1}{2}R$ with R still given by (28).

Improvements on (28) can be obtained by retaining early terms in the series (24). Thus the first term to deviate appreciably from zero is the term in Γ_3. If we retain it, (25) is augmented to

$$A(1+y) + B + C(y^2 + 2y) = \int_{-1}^{1} \frac{F(x)}{x-y}\,dx, \qquad (29)$$

where

$$C = \Delta_3 \int_0^{\pi} E'(\theta)\sin(3\theta/2)\,d\theta, \qquad \Delta_3 = \frac{2a\Gamma_3}{3\pi} - 1.$$

The appropriate solution of (29) is

$$F(x) = \frac{1}{\pi\sqrt{1-x^2}}\left\{Cx^3 + (A+2C)x^2 + (A+B-\tfrac{1}{2}C)x - (\tfrac{1}{2}A + C)\right\}. \qquad (30)$$

If this is substituted in (20), and also into the expressions defining A, B and C, and the latter eliminated, an equation analogous to (28) appears, in which the small quantity $d = 3\Delta_3/(64+28\Delta_3)$ indicates the order of departure from (28)

$$\frac{1+R}{1-R} = j\frac{k'a}{\pi}\,\frac{35\pi + j2K'a - d(119\pi + j18K'a)}{13\pi + j30K'a + d(135\pi - j14K'a)}\,. \qquad (31)$$

It is a straightforward matter to include different dielectric media in the solution. Thus, if the guides on the left have a dielectric constant ϵ, instead of the value unity so far assumed, (16) and (17) hold except that k' becomes $(2\pi/\lambda)\sqrt{\epsilon - (\lambda/2a)^2}$ and γ_n becomes $\sqrt{n^2\pi^2/a^2 - k^2\epsilon}$. The analysis is otherwise unaltered, and the results (28) or (31) are valid with the new values of the constants. Similarly, the dielectric on the right can be varied.

Another configuration which is similarly solvable without much further trouble is the case of the bifurcated waveguide with diaphragms at the junction. In order to maintain symmetry the diaphragm in the lower half is the mirror image of that in the upper. There is no complication, either, to accept arbitrary dielectrics in the different guides.

The effect of the diaphragm first appears in (20), wherein the limits of integration become as in Section V. The solution leads directly to (25), but with the altered limits. The change of variable of (7) then restores the range to -1, 1 and the solution proceeds as before. We shall not, however, pursue the matter any further here, as the example given in Section V is typical of the method of solution obtained.

VII. Inductive Diaphragm and Strip

As a further example we shall examine an obstacle in a rectangular waveguide consisting of the combination of a symmetrical diaphragm with a central strip. Only the inductive case will be examined, as the capacitive case is obtainable from the known result[2] for an unsymmetrical capacitive strip, by the method of images.

The arrangement is shown in Fig. 4. It is apparent that there is a double aperture and a triple metallic obstacle, either method of description being permissible. In order to reduce the problem to one involving a single unknown function we note that, because of the symmetry, a single aperture distribution function suffices,

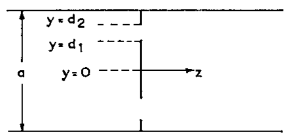

Fig. 4—Inductive diaphragm and strip in rectangular waveguide.

the same for each aperture. Hence we set up the equations from this point of view. As in the previous section, we could allow for different dielectric materials on either side of the discontinuity, but this is an unnecessary complication which will be omitted here.

The field to the left of the obstacle can be written

$$E_x = (e^{-jk'z} + Re^{jk'z})\cos(\pi y/a) + \sum_{1}^{\infty} R_{2n+1}e^{\gamma_{2n+1}z}\cos(\overline{2n+1}\,\pi y/a) \qquad (32)$$

$$Z_0H_yk = k'(e^{-jk'z} - Re^{jk'z})\cos(\pi y/a) + j\sum_{1}^{\infty} R_{2n+1}\gamma_{2n+1}e^{\gamma_{2n+1}z}\cos(\overline{2n+1}\,\pi y/a), \qquad (33)$$

with

$$\gamma_{2n+1} = \sqrt{(2n+1)^2\pi^2/a^2 - k^2} \sim (2n+1)\pi/a \text{ for large } n.$$

To the right a similar form is obtained, except that the first term for E_x is $(1+R)e^{-jk'z}\cos(\pi y/a)$, the magnitude coming from the equality of E_x on the two sides of the boundary. The sign of z in the exponentials is reversed, and this changes the sign of j in the series for H_y. Only odd-order modes appear in the various summations on account of the symmetry of the arrangement.

If the field in the upper aperture is represented by $E(\pi y/a)$, then Fourier analysis gives

$$1 + R = \frac{4}{a} \int_{d_1}^{d_2} \cos(\pi y/a) E(\pi y/a) dy$$
$$= -\frac{4}{\pi} \int_{\pi d_1/a}^{\pi d_2/a} E'(\theta) \sin\theta d\theta \quad (34)$$

on changing variable and integrating by parts. As previously, the integrated part vanishes at the limit because of the physical requirements on E at the edges of the metal inserts.

Similarly for the higher mode coefficients

$$R_{2n+1} = -\frac{4}{\pi} \frac{1}{2n+1} \int_{\pi d_1/a}^{\pi d_2/a} E'(\theta) \sin(\overline{2n+1}\,\theta) d\theta.$$

The equation resulting from equating the tangential magnetic fields on the two sides of the boundary can be written

$$ak'R\cos\phi = -4j \sum_1^\infty \frac{a\gamma_{2n+1}}{\pi(2n+1)} \int E'(\theta) \sin(\overline{2n+1}\,\theta) \sin(\overline{2n+1}\,\phi) d\theta, \quad (35)$$

where the integration and the range of the variable $\phi = \pi y/a$ is over the upper aperture $\pi d_1/a \leq \phi, \theta \leq \pi d_2/a$. From the symmetry of the problem correct conditions are maintained in the lower half of the guide. Now

$$\frac{a\gamma_{2n+1}}{\pi(2n+1)} = 1 - \delta_{2n+1},$$

where $\delta \to 0$ for large n. For the quasi-static solution we ignore δ completely (as previously, the first few terms could be retained to give an improved solution).

On adding the first term of the series to each side, summing and simplifying, (35) can be written

$$A\cos\phi = \int_{\pi d_1/a}^{\pi d_2/a} \frac{E'(\theta)\sin\theta\cos\phi d\theta}{\cos 2\phi - \cos 2\theta}, \quad (36)$$

where

$$A = jak'R/4 + \int_{\pi d_1/a}^{\pi d_2/a} E'(\theta)\sin\theta d\theta. \quad (37)$$

To solve this equation, put $\cos 2\phi = \alpha + \beta x$, $\cos 2\theta = \alpha + \beta y$ and $E'(\theta)\sin\theta d\theta = \beta F(y) dy$. The factor $\cos\phi$ cancels and (36) becomes

$$A = \int_{-1}^{1} \frac{F(y)dy}{y - x}, \text{ provided that}$$

$$\alpha + \beta = \cos(2\pi d_1/a) \text{ and } \alpha - \beta = \cos(2\pi d_2/a). \quad (38)$$

The solution of the integral equation is

$$F(y) = \frac{A}{\pi\sqrt{1 - y^2}}(y + C). \quad (39)$$

In order to determine C we note that, from the vanishing of $E(\theta)$ at the limits,

$$0 = \int_{\pi d_1/a}^{\pi d_2/a} E'(\theta) d\theta = -\beta \int_{-1}^{1} \frac{F(y)dy}{\sin\theta}, \text{ hence}$$

$$\int_{-1}^{1} \frac{y + C}{\sqrt{1 - y^2}} \frac{dy}{\sqrt{1 - \alpha - \beta y}} = 0. \quad (40)$$

The determination of C from this relation is given in Appendix I.

From (34) and (37) we can now calculate the reflection at the aperture, and hence the normalized reactance X representing the discontinuity. In terms of C we have

$$X = -\frac{a}{\lambda_g} \frac{\beta C}{1 + \beta C}.$$

This relation can be put in various forms. Perhaps the simplest is

$$X = \frac{a}{\lambda_g}\left\{-1 + \frac{K}{2\sin^2(\pi d_2/a)E + [\cos^2(\pi d_2/a) - \sin^2(\pi d_1/a)]K}\right\}, \quad (41)$$

where the modulus of the complete elliptic functions K and E is given by

$$k = \sqrt{1 - \sin^2(\pi d_1/a)\operatorname{cosec}^2(\pi d_2/a)}. \quad (42)$$

This may be compared to Lewin,[5] (p. 62) to which it reduces when $d_2 = \frac{1}{2}a$.

VIII. Unsymmetrical H-Plane Step

Fig. 5 shows a waveguide filled with a medium ϵ_1, μ_1 (relative values) from $z = -\infty$ to $z = 0$. At $z = 0$ the guide side $y = a$ is stepped to give a guide of width $2a$ for $z > 0$. This region is filled with medium ϵ_2, μ_2. A wave with electric field $E_x = e^{-jk'z}\sin(\pi y/a)$ is incident from the left, with the propagation constant given by

$$k' = k(\epsilon_1\mu_1 - \lambda^2/4a^2)^{1/2}. \quad (43)$$

The modes which can propagate for $z > 0$ depend on the values of ϵ_2 and μ_2. In particular, if both media are the same, at least two modes can propagate in the wider guide, the dominant mode and the second. They are of the form, respectively,

$$e^{-jK'z}\sin(\pi y/2a) \quad \text{and} \quad e^{-jK_2 z}\sin(\pi y/a),$$

with

$$K' = k(\epsilon_2\mu_2 - \lambda^2/16a^2)^{1/2}$$

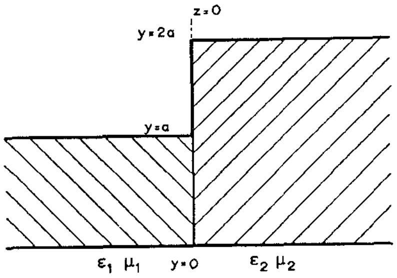

Fig. 5—Change of cross section and of medium in H-plane of rectangular waveguide.

and

$$K_2 = k(\epsilon_2\mu_2 - \lambda^2/4a^2)^{1/2}. \quad (44)$$

When the media are the same, K_2 is the same as k'.

In order to treat the most general case, the loading to these two modes at $z>0$ must be considered. If the second guide is completely matched, there is no reflection in either mode. In general there will be reflections of amplitude R_1' and R_2', with phases dependent on the positioning of the reflecting loads. If these loads are referred to the plane $z=0$, we can define admittances

$$Y_1 = (1 - R_1')/(1 + R_1')$$

and

$$Y_2 = (1 - R_2')/(1 + R_2'), \quad (45)$$

which determine the effects of the reflections. These admittances are relative to the wave-admittances in the second guide, the relevant admittance being that of the respective mode.

The equations for the electric and magnetic fields can now be set up. For $z<0$ we have

$$E_x = (e^{-jk'z} + Re^{jk'z}) \sin(\pi y/a) + \sum_2^\infty R_n e^{\gamma_n z} \sin(n\pi y/a) \quad (46)$$

$$\mu_1 k Z_0 H_y = k'(e^{-jk'z} - Re^{jk'z}) \sin(\pi y/a) + j\sum_2^\infty R_n\gamma_n e^{\gamma_n z} \sin(n\pi y/a), \quad (47)$$

where

$$\gamma_n = (n^2\pi^2/a^2 - k^2\epsilon_1\mu_1)^{1/2} \sim n\pi/a \text{ for large } n. \quad (48)$$

For $z>0$,

$$E_x = \frac{T_1}{1+R_1'}(e^{-jK'z} + R_1'e^{jK'z}) \sin(\pi y/2a) + \frac{T_2}{1+R_2'}(e^{-jK_2z} + R_2'e^{jK_2z}) \sin(\pi y/a) + \sum_3^\infty T_n e^{-\Gamma_n z} \sin(n\pi y/2a) \quad (49)$$

$$\mu_2 k Z_0 H_y = \frac{K'T_1}{1+R_1'}(e^{-jK'z} - R_1'e^{jK'z}) \sin(\pi y/2a) + \frac{K_2T_2}{1+R_2'}(e^{-jK_2z} - R_2'e^{jK_2z}) \sin(\pi y/a) - j\sum_3^\infty T_n\Gamma_n e^{-\Gamma_n z} \sin(n\pi y/2a), \quad (50)$$

where

$$\Gamma_n = (n^2\pi^2/4a^2 - k^2\epsilon_2\mu_2)^{1/2} \sim n\pi/2a \text{ for large } n. \quad (51)$$

In putting the equations in this form the coefficients of the first two modes have been written so as to exhibit fully the effect of the loading in the second guide. Thus, T_1 alone is simply the transmission coefficient in the absence of mismatch in the dominant mode.

At $z=0$ the electric field is taken to be $E(\pi y/a)$ for $0<y<a$, and 0 for $a<y<2a$. At $y=0$, a, it is zero at the metal walls. Otherwise the form of E is as yet undetermined. It will be convenient to take a dummy variable of integration η instead of y, and also to change variables from $\pi\eta/a$ to θ and $\pi y/a$ to ϕ. Then the various coefficients can be determined by Fourier analysis in terms of the as yet unknown E. For example,

$$1 + R = \frac{2}{a}\int_0^a E(\pi\eta/a) \sin(\pi\eta/a)d\eta.$$

If we integrate by parts, taking the integrated part zero because of the vanishing of E at the limits, and change from η to θ as explained above, we get

$$1 + R = \frac{2}{\pi}\int_0^\pi E'(\theta) \cos\theta d\theta. \quad (52)$$

Similarly

$$R_n = \frac{2}{\pi n}\int_0^\pi E'(\theta) \cos n\theta d\theta \quad (53)$$

and

$$T_n = \frac{2}{\pi n}\int_0^\pi E'(\theta) \cos \tfrac{1}{2}n\theta d\theta. \quad (54)$$

Inserting these values in the equations for the magnetic field, putting $z=0$, and equating the two fields over the aperture, gives

$$\tfrac{1}{2}\pi\mu_2 k'(1 - R) \sin\phi + j\sum_2^\infty \mu_2(\gamma_n/n)\int_0^\pi E'(\theta) \cos n\theta d\theta \cdot \sin n\phi$$
$$= \mu_1 K' Y_1 \sin\tfrac{1}{2}\phi \int_0^\pi E'(\theta) \cos\tfrac{1}{2}\theta d\theta + \tfrac{1}{2}\mu_1 K_2 Y_2 \sin\phi \int_0^\pi E'(\theta)\cos\theta d\theta - j\sum_3^\infty \mu_1(\Gamma_n/n)\int_0^\pi E'(\theta) \cos\tfrac{1}{2}n\theta d\theta \cdot \sin\tfrac{1}{2}n\phi. \quad (55)$$

In order to obtain the quasi-static equation we now replace γ_n by $n\pi/a$ and Γ_n by $n\pi/2a$. If desired, a finite number of early terms could be retained in their exact form: this has, in fact, been done with the second-order mode, which happens, in this problem, to be a propagating mode. The series are then extended down to $n=1$ by adding and subtracting terms, and summed using the formula

$$\sum_1^\infty \cos(nu)\sin(nv) = \tfrac{1}{2}\sin v/(\cos u - \cos v).$$

We introduce two constants as follows:

$$A = (j4/\pi^2)(j\pi + 2K'aY_1)\int_0^\pi E'(\theta)\cos\tfrac{1}{2}\theta d\theta \tag{56}$$

$$B = (-j8/\pi)[k'a(1-R)\mu_2/\mu_1 - \tfrac{1}{2}(1+R)(j\pi + K_2aY_2 + j\pi\mu_2/\mu_1)]. \tag{57}$$

Then (55) becomes, after expressing the trigonometrical terms in terms of their half angles,

$$\frac{1}{\pi}\int_0^\pi E'(\theta)\left[\frac{1}{\sin\frac{1}{2}\theta - \sin\frac{1}{2}\phi} - \frac{1}{\sin\frac{1}{2}\theta + \sin\frac{1}{2}\phi}\right]\cdot[\cos\tfrac{1}{2}\theta + \cos\tfrac{1}{2}\phi(1+2\mu_2/\mu_1)]d\theta = A\sin\tfrac{1}{2}\phi + B\sin\tfrac{1}{2}\phi\cos\tfrac{1}{2}\phi \qquad 0<\phi<\pi. \tag{58}$$

Now $E(\theta)$, qua function of θ can be considered to be an odd function. It goes to zero linearly at $\theta=0$. In fact, the alternative problem of a symmetrical waveguide step, fed by the *second*-order mode, and obviously having antisymmetrical features, is converted into the present one by placing a metal wall along the center where the electric field is null. Thus (58), which is obtained only for $0<\phi<\pi$, is in fact still valid for $-\pi<\phi<0$. This can also be seen by putting $-\phi$ for ϕ and $-\theta$ for θ, when the equation is seen to transform into itself, on using the symmetry properties of $E(\theta)$. In order to see this, and at the same time simplify the equation, we note that, if in the second term only, on the left, we write $-\theta$ for θ, it takes the form of the first term, but with limits 0 and $-\pi$. Hence, the left hand side becomes

$$\frac{1}{\pi}\int_{-\pi}^\pi E'(\theta)\frac{\cos\frac{1}{2}\theta + \cos\frac{1}{2}\phi(1+2\mu_2/\mu_1)}{\sin\frac{1}{2}\theta - \sin\frac{1}{2}\phi}d\theta,$$

which obviously has the symmetry properties stated.

Finally we change variable again, putting $x=\sin\frac{1}{2}\theta$ and $y=\sin\frac{1}{2}\phi$. (This use of x and y in this section will not be confused with their earlier use as coordinates.) Instead of $E'(\theta)$ we introduce a function $F(x)$ such that $E'(\theta)d\theta = F(x)dx$.

The quantity $1+2\mu_2/\mu_1$ which occurs repeatedly from here on will be denoted by α^2

$$\alpha^2 = 1 + 2\mu_2/\mu_1 (=3 \text{ for equal media}). \tag{59}$$

Eq. (58) becomes

$$\frac{1}{\pi}\int_{-1}^1 \frac{F(x)}{x-y}[(1-x^2)^{1/2} + \alpha^2(1-y^2)^{1/2}]dx = Ay + By(1-y^2)^{1/2} \qquad -1<y<1, \tag{60}$$

with the additional requirement

$$\int_0^1 F(x)dx = \int_0^\pi E'(\theta)d\theta = E(\pi) - E(0) = 0. \tag{61}$$

Eq. (60) is considerably more involved than the simple singular integral equation which it at first sight appears to be. If the term in α^2 were absent it would be a straightforward equation with $(1-x^2)^{1/2}F(x)$ as unknown. Similarly, if the term in $(1-x^2)^{1/2}$ were absent it would become, on dividing by $(1-y^2)^{1/2}$, a simple equation in $F(x)$. The *general* equation, in which two arbitrary functions appear, would seem to be not solvable by known techniques. However, the particular case (60), and quite a range of other equations, can be solved by a technique outlined in Appendix II. We will here quote only the final results. The solution to (60) which also satisfies (61) is

$$F(x) = \frac{A\sec\pi\beta}{2(1+\alpha^2)}\left[X^\beta\left(\frac{2\beta}{1-x} - 1\right) + X^{-\beta}\left(\frac{2\beta}{1+x} - 1\right)\right] - \frac{B\,\mathrm{cosec}\,\pi\beta}{2(1+\alpha^2)}\left[X^\beta\left(x + 2\beta - \frac{2\beta^2}{1-x}\right) + X^{-\beta}\left(-x + 2\beta - \frac{2\beta^2}{1+x}\right)\right], \tag{62}$$

where

$$X = (1-x)/(1+x)$$

and

$$\beta = \frac{1}{\pi}\tan^{-1}(\alpha)\ (=1/3 \text{ for equal media}). \tag{63}$$

A relation between the constants A and B can now be obtained by insertion into (56),

$$A = \frac{4B}{3\alpha}\,\frac{\beta(1-\beta)(1-2\beta)(1-j2K'aY_1/\pi)}{1-(1-2\beta)^2(1-j2K'aY_1/\pi)}. \tag{64}$$

Finally, if these results are inserted into (52) and (57), an equation results for $(1-R)/(1+R)$, the input admittance relative to the wave admittance of the first waveguide

$$Y_{\text{in}} = \frac{1}{2}\frac{\mu_1}{\mu_2}\frac{1}{k'a}\left\{Y_2K_2a + j\pi\alpha^2\left[1 - \frac{9}{16\beta^2(1-\beta)^2}\frac{1-(1-2\beta)^2(1-j2K'aY_1/\pi)}{9-(1-2\beta)^2(1-j2K'aY_1/\pi)}\right]\right\}. \quad (65)$$

In the case $\mu_2 \gg \mu_1$ this gives $\beta \to \frac{1}{2}$ and $Y_{\text{in}} \sim 0$, an open circuit, as is to be expected.

Eq. (62) can be integrated to give the electric field across the aperture, since

$$E(\theta) = \int_0^\theta E'(\theta)d\theta = \int_0^x F(x)dx.$$

$$E(\theta) = \frac{A \sec \pi\beta}{1+\alpha^2}\left[\frac{X^{1-\beta} - X^{\beta}}{1+X}\right] - \frac{B \operatorname{cosec} \pi\beta}{1+\alpha^2}\left[\frac{X^{1-\beta} - X^{1+\beta}}{(1+X)^2} - \beta\frac{X^{1-\beta} - X^{\beta}}{1+X}\right]. \quad (66)$$

Herein we must take $x = \sin \frac{1}{2}\theta$ giving $X = \tan^2 (\overline{\pi - \theta/4})$.

Now, for equal media we have $\beta = \frac{1}{3}$, whilst for $\mu_2 \gg \mu_1$, $\beta = \frac{1}{2}$. β is always the smallest of the various exponents of X. Hence, near the sharp corner, where $\theta \to \pi$, the electric field is seen to vary as X^β or $(\pi - \theta)^{2\beta}$. This can now be expressed in terms of the coordinate y across the aperture,

$$E \sim (a - y)^{2\beta} \quad \text{as} \quad y \to a. \quad (67)$$

Expressing the exponent in terms of α through (63), and hence in terms of μ_1 and μ_2, we get

$$2\beta = \frac{2}{\pi} \tan^{-1} (1 + 2\mu_2/\mu_1)^{1/2}. \quad (68)$$

For equal media this exponent is $\frac{2}{3}$. In prior calculations of this sort of problem it has been usual to conformally map the boundary by the Schwartz-Christoffel transformation, and the exponent has arisen from a consideration of the geometry of the boundary surface at the re-entrant corner. Eq. (68) on the other hand, exhibits the exponent in terms of electrical parameters with a value which varies according to the changes in the media. This result appears to be new. In particular, with $\mu_2 \gg \mu_1$, the first guide is open-circuited, and (68) gives $2\beta = 1$, corresponding to the normal sinusoidal waveguide mode going to zero linearly at the guide wall.

IX. Range of Applicability

The present method is applicable, so far as is known, only to rectangular waveguides, including infinite parallel plate arrangements. When the obstacle is inductive a change of dielectric constant on either side of the boundary can be accommodated. Similarly, when the obstacle is capacitive a change in permeability of the media can be met. But the analysis may sometimes be more involved, apparently, the other way round. The method can be used for any finite number of feeding modes, and can be used to calculate the electric fields, the reflection coefficient, transmission coefficient, or the mode conversion. The quasi-static solution can be augmented by the retention of a finite number of higher-order modes, the solution of the relevant integral equation being not appreciably more complicated thereby. However, each new mode introduces an additional constant into the solution, whose elimination is complicated if too many modes are retained.

The method has not yet been successfully applied in those cases in which guides of different dimensions are involved, except where integrally related, nor in cases in which an axial extension of an obstacle needs to be taken into account. It is believed that at least the first limitation may be eventually removed, but success will depend on new methods of dealing with the special types of singular integral equations that arise.

Appendix I

In (40) put $y = -\cos 2\theta$.

$$0 = \int_0^{\pi/2} \frac{C - 1 + 2\sin^2\theta}{\sqrt{1 - \alpha + \beta - 2\beta \sin^2\theta}} d\theta.$$

Multiply through by $\sqrt{1-\alpha+\beta}$ and put $k^2 = 2\beta/(1-\alpha+\beta)$.

$$0 = \int_0^{\pi/2} \frac{(C - 1 + 2/k^2) - (1 - k^2\sin^2\theta)2/k^2}{\sqrt{1 - k^2 \sin^2\theta}} d\theta$$

where

$$C = 1 + \frac{E - K}{\frac{1}{2}Kk^2} \bmod \sqrt{2\beta/(1 - \alpha + \beta)}.$$

On simplifying the various terms this leads to (41) and (42) of the text.

Appendix II

Following Tricomi[6] we introduce the transform operator T_y operating on a function $\phi(x)$

$$T_y[\phi(x)] = \frac{1}{\pi}\int_{-1}^{1} \frac{\phi(x)}{x - y} dx. \quad (69)$$

The principal value of the integral is to be understood. Where no confusion of the variables is likely to exist we shall write this simply $T(\phi)$. Eq. (60) is a particular case of the equation

$$aT(F) + T(bF) = f, \quad (70)$$

with

$$a = \alpha^2(1 - y^2)^{1/2}, \qquad b = (1 - x^2)^{1/2},$$
$$f = Ay + By(1 - y^2)^{1/2} \tag{71}$$

and subject to $0 = \int_{-1}^{1} F(x)dx$, an alternative form of (61) for $F(x)$ even. The similar problem, but with the addition of inductive diaphragms, would give, after renormalization, a like equation, but with the radicals replaced by the form $[1-(c+dy)^2]^{1/2}$ throughout. In the absence of a general solution of (70), this latter problem so far remains unsolved. Eq. (70) can, however, be solved in a large number of special cases, including $a = b$, $ab = C(1 - x^2)$ and many others. But we shall here concentrate only on the analysis which leads to the solution of the problem in hand.

Tricomi gives a number of useful results. A convolution theorem is

$$T[\phi_1 T(\phi_2) + \phi_2 T(\phi_1)] = T(\phi_1)T(\phi_2) - \phi_1\phi_2. \tag{72}$$

The solution of $T(\phi) = \psi$ is

$$\phi = (1 - y^2)^{-1/2}\{-T_y[(1 - x^2)^{1/2}\psi(x)] + C\}, \tag{73}$$

where C is a constant.

The solution of Carleman's equation

$$a(x)\phi(x) - T_x[\phi(y)] = g(x)$$

is

$$\phi(y) = \frac{a(y)g(y)}{1 + a^2(y)} + A(y)\left\{T_y\left[\frac{e^{-\tau(x)}g(x)}{[1 + a^2(x)]^{1/2}}\right] + \frac{C}{1 - x}\right\}, \tag{74}$$

where

$$\tau(x) = T_x\,[\tan^{-1} 1/a(y)] \quad (0, \pi)$$

and

$$A(x) = e^{\tau(x)}[1 + a^2(x)]^{-1/2}$$

Using these formulas we return to (70) and ask under what circumstances, if at all, it can be reduced to an example of Carleman's equation, whose solution we know.

Let us start with the equation

$$cF + eT(dF) = h, \tag{75}$$

where c, d, e and h are all functionals at our disposal. Operating with T, and using (72) to express $T[eT(dF)]$ in other terms, we get

$$T(h) = T(cF) - edF + T(e)T(dF) - T[dFT(e)]. \tag{76}$$

Let us choose $T(e) = K$, a constant, so that, from (73)

$$e = K(x - C)(1 - x^2)^{-1/2}, \tag{77}$$

with C as yet arbitrary.

Then (76) simplifies to

$$T(h) = T(cF) - edF. \tag{78}$$

From (75) and (78), on eliminating the untransformed terms in F,

$$T(h) + edh/c = T(cF) + (e^2d/c)T(dF). \tag{79}$$

Choose $c = x - C$, $d = (1 - x^2)^{1/2}$, so that $ed/c = K$. Now

$$T(cF) = \frac{1}{\pi}\int_{-1}^{1} \frac{x - C}{x - y} F(x)dx = \frac{1}{\pi}\int_{-1}^{1} \frac{x - y + y - C}{x - y} F(x)dx = (y - C)T(F) \text{ since } \int_{-1}^{1} F(x)dx = 0 \text{ by (61)}.$$

Putting $K = 1/\alpha$, and substituting these results in (79) gives finally

$$\alpha^2(1 - y^2)^{1/2}T(F) + T[(1 - x^2)^{1/2}F] = \frac{\alpha^2(1 - y^2)^{1/2}}{y - C}[T(h) + h/\alpha]. \tag{80}$$

If now we choose h such that

$$T(h) + h/\alpha = \alpha^{-2}(y - C)(1 - y^2)^{-1/2}f(y), \tag{81}$$

then (80) becomes equivalent to (70) and (71).

From (78) with h given by (81)

$$T[(x - C)F] - (y - C)F/\alpha = \tfrac{1}{2}[T(h) - h/\alpha] + \tfrac{1}{2}\alpha^{-2}(y - C)(1 - y^2)^{-1/2}f(y).$$

If we put $(y - C)F(y) = \frac{1}{2}H(y) + \frac{1}{2}h(y)$, then H is given by

$$T(H) - H/\alpha = \alpha^{-2}(y - C)(1 - y^2)^{-1/2}f(y), \tag{82}$$

an equation differing from (81) for h only by the sign of α. These two equations, both of Carleman's type, can now be solved for h and H, and $F(y)$ is given in terms of them by

$$F(y) = \frac{H(y) + h(y)}{2(y - C)}. \tag{83}$$

In writing down the solution it becomes apparent that the integration diverges at $x = 1$ unless the so far arbitrary constant C is given the value unity. This feature determines the constant, and the solution takes the form

$$F(y) = \frac{1}{2\pi(1 - y)(1 + \alpha^2)}\left\{Y^{1-\beta}\int_{-1}^{1} \frac{X^{\beta-1/2}}{y - x} f(x)dx + Y^{\beta}\int_{-1}^{1} \frac{X^{1/2-\beta}}{y - x} f(x)dx\right\}, \tag{84}$$

where $Y = (1 - y)/(1 + y)$ and X is a similar function of x. The constant β is related to α by $\beta = (1/\pi)\tan^{-1}\alpha$.

In view of the requirement of the integrability of $F(y)$ at $y=1$ no complementary functions along the lines indicated in (74) appear in this problem.

The integrations in (84) are straightforward when X is taken as a new variable. If we write

$$y_n = \int_0^\infty X^p(1+X)^{-n}dX,$$

then $y_1 = -\pi$ cosec πp, and the recurrence relation $y_n = y_{n-1}(n-2-p)/(n-1)$ is readily obtained. The integral

$$\int_0^\infty \frac{X^p}{X-Y}dX = -Y^p\pi \cot \pi p$$

on taking $\xi = X/Y$ as new variable. These results suffice to integrate (84) when $f(x)$ is of the form specified in (71) and p is suitably chosen.

It is readily verified that $\int_{-1}^{1} F(x)dx$ is zero, confirming the absence of any other complementary function.

The integrations involved in the determination of A and $1+R$ involve the integrals

$$x_n = \int_0^\infty X^p(1-X)^n(1+X)^{-n-2}dX$$

which can be shown to satisfy

$$(n+1)x_n + 2px_{n-1} - (n-1)x_{n-2} = 0$$

x_0 and x_1 are readily expressed in terms of y_2 and y_3 of the previous paragraph, whence the value of x_n follows.

Finally, (62) of the text for $F(x)$ is found on collection of terms, Its integration to give $E(\theta)$ is elementary.

The solution to (70) with completely arbitrary functions $a(x)$ and $b(x)$ has not yet been determined by this method, and it is possible that in this general case more powerful mathematical tools are required. In particular, it has not been possible to find the solution when the radical takes the more complicated form appropriate to the presence of a diaphragm, except in the very special case $\alpha = 1(\mu_2 = 0)$.

A Dielectric Surface-Wave Structure: the V-Line*

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Summary—**Properties of the V-line, a wedge-shaped surface-wave structure comprising a cylindrical dielectric binding medium of sectorial cross section supported by two conducting plates, are considered in terms of its higher-order hybrid modes of propagation. Practical modifications of the ideal structure are emphasized.**

Design curves and equations are presented to determine various propagation parameters and their significance is discussed. Experimental verification of the theory is described.

Introduction

An analysis of surface-wave propagation on dielectric cylinders of sectorial cross section bounded by conducting plates, as in Fig. 1, leads to the usual set of low-order transverse modes and higher-order hybrid modes. In cases of practical interest, however, the prototype structure, here designated "V-line," will be modified in that the plates will be insulated at the apex, whereupon the transverse modes are eliminated and consideration of high-order hybrid modes is required. This modification of the V-line enhances its versatility; in particular, it facilitates the excitation of the modes and permits the application of biasing potentials between the conducting plates. With the use of ferroelectric cylinders, such bias fields may provide convenient electronic control of propagation characteristics.

Although the angle included by the plates is, in principle, unrestricted, for simplicity it will be taken to be an aliquot portion of a semicircle, *i.e.*, π/n radians, where the integer n designates the order of the mode. For such angles, the modes that may be supported by the V-line may propagate on full circular dielectric cylinders as well. The latter waveguide has undergone extensive analysis with respect to its dominant modes.[1–6]

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[1] R. E. Beam, *et al.*, "Dielectric Tube Waveguides," Northwestern University, Evanston, Ill., Report A.T.I. 94929, ch. 5; 1950.

[2] S. A. Schelkunoff, "Electromagnetic Waves," D. Van Nostrand Co., Inc., New York, N. Y., p. 427; 1943.

[3] A. Sommerfeld, "Electrodynamics," Academic Press, Inc., New York, N. Y., pp. 177–193; 1952.

[4] H. Wegener, "Ausbreitungsgeschwindigkeit, Wellenwiderstand, und Dampfung electromagnetischer Wellen an dielektrischen Zylindern," Forschungbericht Nr. 2018, Deutsch Luftfahrtforschung. Vierjahresplan-Inst. für Schwingungsforschung, Berlin, Germany; August 26, 1944. (CADO Wright-Patterson AF Base, Dayton, Ohio, Document No. ZWB/FB/Re/2018.)

[5] C. H. Chandler, "An investigation of dielectric rod as waveguide," *J. Appl. Phys.*, vol. 20, pp. 1188–1192; December, 1949.

[6] W. M. Elsasser, "Attenuation in a dielectric circular rod," *J. Appl. Phys.*, vol. 20, pp. 1193–1196; December, 1949.